Introduction to perturbation theory 10

Perturbation theory consists of a very useful collection of methods for finding approximate solutions of "perturbed" problems which are close to completely solvable "non-perturbed" problems. These methods can be easily justified if we are investigating motion over a small interval of time. Relatively little is known about how far we can trust the conclusions of perturbation theory in investigating motion over large or infinite intervals of time.

We will see that the motion in many "non-perturbed" integrable problems turns out to be conditionally periodic. In the study of unperturbed problems, and even more so in the study of the perturbed problems, special symplectic coordinates, called "action-angle" variables, are useful. In conclusion, we will prove a theorem justifying perturbation theory for single-frequency systems and will prove the adiabatic invariance of action variables in such systems.

49 Integrable systems

In order to integrate a system of $2n$ ordinary differential equations, we must know $2n$ first integrals. It turns out that if we are given a canonical system of differential equations, it is often sufficient to know only n first integrals—each of them allows us to reduce the order of the system not just by one, but by two.

A *Liouville's theorem on integrable systems*

Recall that a function F is a first integral of a system with hamiltonian function H if and only if the Poisson bracket

$$(H, F) \equiv 0$$

is identically equal to zero.

Definition. Two functions F_1 and F_2 on a symplectic manifold are *in involution* if their Poisson bracket is equal to zero.

Liouville proved that if, in a system with n degrees of freedom (i.e., with a $2n$-dimensional phase space), n independent first integrals in involution are known, then the system is integrable by quadratures.

Here is the exact formulation of this theorem: Suppose that we are given n functions in involution on a symplectic $2n$-dimensional manifold

$$F_1, \ldots, F_n \qquad (F_i, F_j) \equiv 0, \qquad i, j = 1, 2, \ldots, n.$$

Consider a level set of the functions F_i

$$M_{\mathbf{f}} = \{x : F_i(x) = f_i, i = 1, \ldots, n\}.$$

Assume that the n functions F_i are independent on $M_{\mathbf{f}}$ (i.e., the n 1-forms dF_i are linearly independent at each point of $M_{\mathbf{f}}$). Then

1. $M_{\mathbf{f}}$ is a smooth manifold, invariant under the phase flow with hamiltonian function $H = F_1$.
2. If the manifold $M_{\mathbf{f}}$ is compact and connected, then it is diffeomorphic to the n-dimensional torus

$$T^n = \{(\varphi_1, \ldots, \varphi_n) \bmod 2\pi\}.$$

3. The phase flow with hamiltonian function H determines a conditionally periodic motion on $M_{\mathbf{f}}$, i.e., in angular coordinates $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ we have

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}).$$

4. The canonical equations with hamiltonian function H can be integrated by quadratures.

Before proving this theorem, we note a few of its corollaries.

Corollary 1. *If, in a canonical system with two degrees of freedom, a first integral F is known which does not depend on the hamiltonian H, then the system is integrable by quadratures; a compact connected two-dimensional submanifold of the phase space $H = h$, $F = f$ is an invariant torus, and motion on it is conditionally periodic.*

PROOF. F and H are in involution since F is a first integral of a system with hamiltonian function H. □

As an example with three degrees of freedom, we consider a heavy symmetric Lagrange top fixed at a point on its axis. Three first integrals are immediately obvious: H, M_z, and M_3. It is easy to verify that the integrals

M_z and M_3 are in involution. Furthermore, the manifold $H = h$ in the phase space is compact. Therefore, we can immediately say, without any calculations, that for the majority of initial conditions[87] the motion of the top is conditionally periodic: the phase trajectories fill up the three-dimensional torus $H = c_1$, $M_z = c_2$, $M_3 = c_3$. The corresponding three frequencies are called frequencies of fundamental rotation, precession, and nutation.

Other examples arise from the following observation: *if a canonical system can be integrated by the method of Hamilton–Jacobi, then it has n first integrals in involution.* The method consists of a canonical transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$ such that the Q_i are first integrals. But the functions Q_i and Q_j are clearly in involution.

In particular, the observation above applies to the problem of attraction by two fixed centers. Other examples are easily found. In fact, the theorem of Liouville formulated above covers all the problems of dynamics which have been integrated to the present day.

B *Beginning of the proof of Liouville's theorem*

We turn now to the proof of the theorem. Consider the level set of the integrals:

$$M_{\mathbf{f}} = \{x : F_i = f_i, i = 1, \ldots, n\}.$$

By hypothesis, the n 1-forms dF_i are linearly independent at each point of $M_{\mathbf{f}}$; therefore, by the implicit function theorem, $M_{\mathbf{f}}$ is an n-dimensional submanifold of the $2n$-dimensional phase space.

Lemma 1. *On the n-dimensional manifold $M_{\mathbf{f}}$ there exist n tangent vector fields which commute with one another and which are linearly independent at every point.*

Proof. The symplectic structure of phase space defines an operator I taking 1-forms to vector fields. This operator I carries the 1-form dF_i to the field $I\, dF_i$ of phase velocities of the system with hamiltonian function F_i. We will show that *the n fields $I\, dF_i$ are tangent to $M_{\mathbf{f}}$, commute, and are independent.*

The independence of the $I\, dF_i$ at every point of $M_{\mathbf{f}}$ follows from the independence of the dF_i and the nonsingularity of the isomorphism I. The fields $I\, dF_i$ commute with one another, since the Poisson brackets of their hamiltonian functions (F_i, F_j) are identically 0. For the same reason, the derivative of the function F_i in the direction of the field $I\, dF_j$ is equal to zero for any $i, j = 1, \ldots, n$. Thus the fields $I\, dF_i$ are tangent to $M_{\mathbf{f}}$, and Lemma 1 is proved. □

[87] The singular level sets, where the integrals are not functionally independent, constitute the exception.

We notice that we have proved even more than Lemma 1:

1′. The manifold $M_{\mathbf{f}}$ is invariant with respect to each of the n commuting phase flows g_i^t with hamiltonian functions $F_i: g_i^t g_j^s = g_j^s g_i^t$.
1″. The manifold $M_{\mathbf{f}}$ is null (i.e., the 2-form ω^2 is zero on $TM_{\mathbf{f}}|_x$).

This is true since the n vectors $I\, dF_i|_x$ are skew-orthogonal to one another ($(F_i, F_j) \equiv 0$) and form a basis of the tangent plane to the manifold $M_{\mathbf{f}}$ at the point x.

C *Manifolds on which the action of the group $\mathbb{R}^n$ is transitive*

We will now use the following topological proposition (the proof is completed in Section D).

Lemma 2. *Let M^n be a compact connected differentiable n-dimensional manifold, on which we are given n pairwise commutative and linearly independent at each point vector fields. Then M^n is diffeomorphic to an n-dimensional torus.*

PROOF. We denote by g_i^t, $i = 1, \ldots, n$, the one-parameter groups of diffeomorphisms of M corresponding to the n given vector fields. Since the fields commute, the groups g_i^t and g_j^s commute. Therefore, we can define an action g of the commutative group $\mathbb{R}^n = \{\mathbf{t}\}$ on the manifold M by setting

$$g^{\mathbf{t}}: M \to M \qquad g^{\mathbf{t}} = g_1^{t_1} \cdots g_n^{t_n}, \qquad (\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n).$$

Clearly, $g^{\mathbf{t}+\mathbf{s}} = g^{\mathbf{t}} g^{\mathbf{s}}$, $\mathbf{t}, \mathbf{s} \in \mathbb{R}^n$. Now fix a point $x_0 \in M$. Then we have a map

$$g: \mathbb{R}^n \to M \qquad g(\mathbf{t}) = g^{\mathbf{t}} x_0.$$

(The point x_0 moves along the trajectory of the first flow for time t_1, along the second flow for time t_2, etc.)

PROBLEM 1. Show that the map g (Figure 211) of a sufficiently small neighborhood V of the point $0 \in \mathbb{R}^n$ gives a chart in a neighborhood of x_0: every point $x_0 \in M$ has a neighborhood U ($x_0 \in U \subset M$) such that g maps V diffeomorphically onto U.

Hint. Apply the implicit function theorem and use the linear independence of the fields at x_0.

PROBLEM 2. Show that $g: \mathbb{R}^n \to M$ is onto.

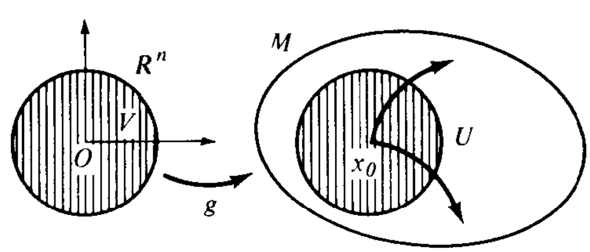

Figure 211 Problem 1

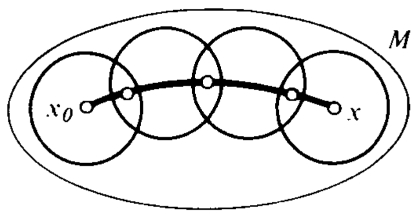

Figure 212 Problem 2

Hint. Connect a point $x \in M$ with x_0 by a curve (Figure 212), cover the curve by a finite number of the neighborhoods U of the preceding problem and define $\mathbf{t}$ as the sum of shifts $\mathbf{t}_i$ corresponding to pieces of the curve.

We note that the map $g: \mathbb{R}^n \to M^n$ cannot be one-to-one since M^n is compact and $\mathbb{R}^n$ is not. We will examine the set of pre-images of $x_0 \in M^n$.

Definition. The *stationary group* of the point x_0 is the set Γ of points $\mathbf{t} \in \mathbb{R}^n$ for which $g^{\mathbf{t}}x_0 = x_0$.

PROBLEM 3. Show that Γ is a subgroup of the group $\mathbb{R}^n$, independent of the point x_0.

Solution. If $g^{\mathbf{s}}x_0 = x_0$ and $g^{\mathbf{t}}x_0 = x_0$, then $g^{\mathbf{s}+\mathbf{t}}x_0 = g^{\mathbf{s}}g^{\mathbf{t}}x_0 = g^{\mathbf{s}}x_0 = x_0$ and $g^{-\mathbf{t}}x_0 = g^{-\mathbf{t}}g^{\mathbf{t}}x_0 = x_0$. Therefore, Γ is a subgroup of $\mathbb{R}^n$. If $x = g^{\mathbf{r}}x_0$ and $\mathbf{t} \in \Gamma$, then $g^{\mathbf{t}}x = g^{\mathbf{t}+\mathbf{r}}x_0 = g^{\mathbf{r}}g^{\mathbf{t}}x_0 = g^{\mathbf{r}}x_0 = x$.

In this way the stationary group Γ is a well-defined subgroup of $\mathbb{R}^n$ independent of the point x_0. In particular, the point $\mathbf{t} = 0$ clearly belongs to Γ.

PROBLEM 4. Show that, in a sufficiently small neighborhood V of the point $0 \in \mathbb{R}^n$, there is no point of the stationary group other than $\mathbf{t} = 0$.

Hint. The map $g: V \to U$ is a diffeomorphism.

PROBLEM 5. Show that, in the neighborhood $\mathbf{t} + V$ of any point $\mathbf{t} \in \Gamma \subset \mathbb{R}^n$, there is no point of the stationary group Γ other than $\mathbf{t}$. (Figure 213)

Thus the points of the stationary group Γ lie in $\mathbb{R}^n$ *discretely*. Such subgroups are called *discrete subgroups*.

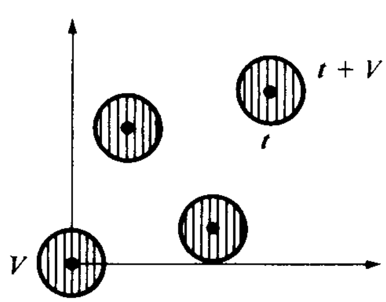

Figure 213 Problem 5

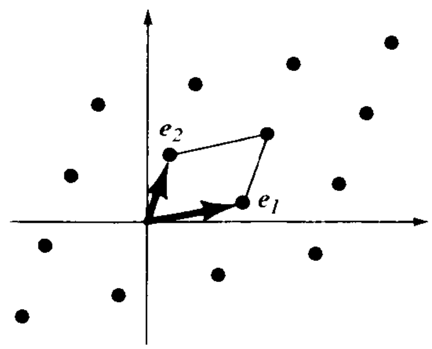

Figure 214 A discrete subgroup of the plane

Example. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be k linearly independent vectors in $\mathbb{R}^n$, $0 \le k \le n$. The set of all their integral linear combinations (Figure 214)

$$m_1\mathbf{e}_1 + \cdots + m_k\mathbf{e}_k, \qquad m_i \in \mathbb{Z} = (\ldots, -2, -1, 0, 1, \ldots)$$

forms a discrete subgroup of $\mathbb{R}^n$. For example, the set of all integral points in the plane is a discrete subgroup of the plane.

D *Discrete subgroups in* $\mathbb{R}^n$

We will now use the algebraic fact that the example above includes all discrete subgroups of $\mathbb{R}^n$. More precisely, we will prove

Lemma 3. *Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist k ($0 \le k \le n$) linearly independent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \Gamma$ such that Γ is exactly the set of all their integral linear combinations.*

Proof. We will consider $\mathbb{R}^n$ with some euclidean structure. We always have $0 \in \Gamma$. If $\Gamma = \{0\}$ the lemma is proved. If not, there is a point $\mathbf{e}_0 \in \Gamma$, $\mathbf{e}_0 \ne 0$ (Figure 215). Consider the line $\mathbb{R}\mathbf{e}_0$. We will show that among the elements of Γ on this line, there is a point $\mathbf{e}_1$ which is closest to 0. In fact, in the disk of radius $|\mathbf{e}_0|$ with center 0, there are only a finite number of points of Γ (as we saw above, every point x of Γ has a neighborhood V of standard size which does not contain any other point of Γ). Among the finite number of points of Γ inside this disc and lying on the line $\mathbb{R}\mathbf{e}_0$, the point closest to 0 will be the closest point to 0 on the whole line. The integral multiples of this point $\mathbf{e}_1$ ($m\mathbf{e}_1$, $m \in \mathbb{Z}$) constitute the intersection of the line $\mathbb{R}\mathbf{e}_0$ with Γ.

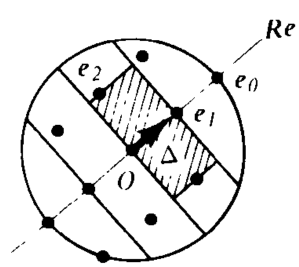

Figure 215 Proof of the lemma on discrete subgroups

In fact, the points $m\mathbf{e}_1$ divide the line into pieces of length $|\mathbf{e}_1|$. If there were a point $\mathbf{e} \in \Gamma$ inside one of these pieces $(m\mathbf{e}_1, (m+1)\mathbf{e}_1)$, then the point $\mathbf{e} - m\mathbf{e}_1 \in \Gamma$ would be closer to 0 than $\mathbf{e}_1$.

If there are no points of Γ off the line $\mathbb{R}\mathbf{e}_1$, the lemma is proved. Suppose there is a point $\mathbf{e} \in \Gamma, \mathbf{e} \notin \mathbb{R}\mathbf{e}_1$. We will show that there is a point $\mathbf{e}_2 \in \Gamma$ closest to the line $\mathbb{R}\mathbf{e}_1$ (but not lying on the line). We project $\mathbf{e}$ orthogonally onto $\mathbb{R}\mathbf{e}_1$. The projection lies in exactly one interval $\Delta = \{\lambda\mathbf{e}_1\}$, $m \leq \lambda < m + 1$. Consider the right circular cylinder C with axis Δ and radius equal to the distance from Δ to $\mathbf{e}$. In this cylinder lie a finite (nonempty) number of points of the group Γ. Let $\mathbf{e}_2$ be the closest one to the axis $\mathbb{R}\mathbf{e}_1$ not lying on the axis.

Problem 6. Show that the distance from this axis to any point $\mathbf{e}$ of Γ not lying on $\mathbb{R}\mathbf{e}_1$ is greater than or equal to the distance of $\mathbf{e}_2$ from $\mathbb{R}\mathbf{e}_1$.

Hint. By a shift of $m\mathbf{e}_1$ we can move the projection of $\mathbf{e}$ onto the axis interval Δ.

The integral linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$ form a lattice in the plane $\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2$.

Problem 7. Show that there are no points of Γ on the plane $\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2$ other than integral linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$.

Hint. Partition the plane into parallelograms (Figure 216) $\Delta = \{\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2\}$, $m_i \leq \lambda_i < m_i + 1$. If there were an $\mathbf{e} \in \Delta$ with $\mathbf{e} \neq m_1\mathbf{e}_1 + m_2\mathbf{e}_2$, then the point $\mathbf{e} - m_1\mathbf{e}_1 - m_2\mathbf{e}_2$ would be closer to $\mathbb{R}\mathbf{e}_1$ than $\mathbf{e}_2$.

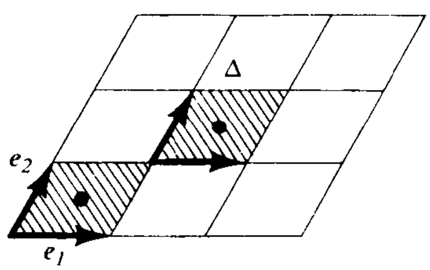

Figure 216 Problem 7

If there are no points of Γ outside the plane $\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2$, the lemma is proved. Suppose that there is a point $\mathbf{e} \in \Gamma$ outside this plane. Then there exists a point $\mathbf{e}_3 \in \Gamma$ closest to $\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2$; the points $m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3$ exhaust Γ in the three-dimensional space $\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2 + \mathbb{R}\mathbf{e}_3$. If Γ is not exhausted by these, we take the closest point to this three-dimensional space, etc.

Problem 8. Show that this closest point always exists.

Hint. Take the closest of the finite number of points in a "cylinder" C.

Note that the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ are linearly independent. Since they all lie in $\mathbb{R}^n$, there are $k \leq n$ of them.

Problem 9. Show that Γ is exhausted by the integral linear combinations of $\mathbf{e}_1, \ldots, \mathbf{e}_k$.

Hint. Partition the plane $\mathbb{R}\mathbf{e}_1 + \cdots + \mathbb{R}\mathbf{e}_k$ into parallelepipeds Δ and show that there cannot be a point of Γ in any Δ. If there is an $\mathbf{e} \in \Gamma$ outside the plane $\mathbb{R}\mathbf{e}_1 + \cdots + \mathbb{R}\mathbf{e}_k$, the construction is not finished.

Thus Lemma 3 is proved. ☐

It is now easy to prove Lemma 2: $M_{\mathbf{f}}$ is diffeomorphic to a torus T^n.

Consider the direct product of k circles and $n - k$ straight lines:

$$T^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots, \varphi_k; y_1, \ldots, y_{n-k})\}, \qquad \boldsymbol{\varphi} \bmod 2\pi,$$

together with the natural map $p\colon \mathbb{R}^{2n} \to T^k \times \mathbb{R}^{n-k}$,

$$p(\boldsymbol{\varphi}, \mathbf{y}) = (\boldsymbol{\varphi} \bmod 2\pi, \mathbf{y}).$$

The points $\mathbf{f}_1, \ldots, \mathbf{f}_k \in \mathbb{R}^n$ ($\mathbf{f}_i$ has coordinates $\varphi_i = 2\pi$, $\varphi_j = 0$, $\mathbf{y} = 0$) are mapped to 0 under this map.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \Gamma \subset \mathbb{R}^n$ be the generators of the group Γ (cf. Lemma 3). We map the vector space $\mathbb{R}^n = \{(\boldsymbol{\varphi}, \mathbf{y})\}$ onto the space $\mathbb{R}^n = \{\mathbf{t}\}$ so that the vectors $\mathbf{f}_i$ go to $\mathbf{e}_i$. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be such an isomorphism.

We now note that $\mathbb{R}^n = \{(\boldsymbol{\varphi}, \mathbf{y})\}$ gives charts for $T^k \times \mathbb{R}^{n-k}$, and $\mathbb{R}^n = \{\mathbf{t}\}$ gives charts for our manifold $M_{\mathbf{f}}$.

Problem 10. Show that the map of charts $A\colon \mathbb{R}^n \to \mathbb{R}^n$ gives a diffeomorphism $\tilde{A}\colon T^k \times \mathbb{R}^{n-k} \to M_{\mathbf{f}}$,

$$\begin{array}{ccc} \mathbb{R}^n = \{(\boldsymbol{\varphi}, \mathbf{y})\} & \xrightarrow{\ A\ } & \mathbb{R}^n = \{\mathbf{t}\} \\ \Big\downarrow p & & \Big\downarrow g \\ T^k \times \mathbb{R}^{n-k} & \xrightarrow{\ \tilde{A}\ } & M_{\mathbf{f}} \end{array}$$

But, since the manifold $M_{\mathbf{f}}$ is compact by hypothesis, $k = n$ and $M_{\mathbf{f}}$ is an n-dimensional torus. Lemma 2 is proved. ☐

In view of Lemma 1, the first two statements of the theorem are proved. At the same time, we have constructed angular coordinates $\varphi_1, \ldots, \varphi_n \bmod 2\pi$ on $M_{\mathbf{f}}$.

Problem 11. Show that under the action of the phase flow with hamiltonian H the angular coordinates $\boldsymbol{\varphi}$ vary uniformly with time

$$\dot{\varphi}_i = \omega_i \qquad \omega_i = \omega_i(\mathbf{f}) \qquad \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}t.$$

In other words, motion on the invariant torus $M_{\mathbf{f}}$ is conditionally periodic.

Hint. $\boldsymbol{\varphi} = A^{-1}\mathbf{t}$.

Of all the assertions of the theorem, only the last remains to be proved: that the system can be integrated by quadratures.

50 Action-angle variables

We show here that, under the hypotheses of Liouville's theorem, we can find symplectic coordinates $(\mathbf{I}, \boldsymbol{\varphi})$ such that the first integrals $\mathbf{F}$ depend only on $\mathbf{I}$, and $\boldsymbol{\varphi}$ are angular coordinates on the torus $M_{\mathbf{f}}$.

A *Description of action-angle variables*

In Section 49 we studied one particular compact connected level manifold of the integrals: $M_{\mathbf{f}} = \{x: \mathbf{F}(x) = \mathbf{f}\}$; it turned out that $M_{\mathbf{f}}$ was an n-dimensional torus, invariant with respect to the phase flow. We chose angular coordinates φ_i on M so that the phase flow with hamiltonian function $H = F_1$ takes an especially simple form:

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}(\mathbf{f}) \qquad \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}t.$$

We will now look at a neighborhood of the n-dimensional manifold $M_{\mathbf{f}}$ in $2n$-dimensional phase space.

PROBLEM. Show that the manifold $M_{\mathbf{f}}$ has a neighborhood diffeomorphic to the direct product of the n-dimensional torus T^n and the disc D^n in n-dimensional euclidean space.

Hint. Take the functions F_i and the angles φ_i constructed above as coordinates. In view of the linear independence of the dF_i, the functions F_i and φ_i $(i = 1, \ldots, n)$ give a diffeomorphism of a neighborhood of $M_{\mathbf{f}}$ onto the direct product $T^n \times D^n$.

In the coordinates $(\mathbf{F}, \boldsymbol{\varphi})$ the phase flow with hamiltonian function $H = F_1$ can be written in the form of the simple system of $2n$ ordinary differential equations

$$\frac{d\mathbf{F}}{dt} = 0 \qquad \frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}(\mathbf{F}), \tag{1}$$

which is easily integrated: $\mathbf{F}(t) = \mathbf{F}(0)$, $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}(\mathbf{F}(0))t$.

Thus, in order to integrate explicitly the original canonical system of differential equations, it is sufficient to find the variables $\boldsymbol{\varphi}$ in explicit form. It turns out that this can be done using only quadratures. A construction of the variables $\boldsymbol{\varphi}$ is given below.

We note that the variables $(\mathbf{F}, \boldsymbol{\varphi})$ are not, in general, symplectic coordinates. It turns out that there are functions of $\mathbf{F}$, which we will denote by $\mathbf{I} = \mathbf{I}(\mathbf{F})$, $\mathbf{I} = (I_1, \ldots, I_n)$, such that the variables $(\mathbf{I}, \boldsymbol{\varphi})$ are symplectic coordinates: the original symplectic structure ω^2 is expressed in them by the usual formula

$$\omega^2 = \sum dI_i \wedge d\varphi_i.$$

The variables $\mathbf{I}$ are called action variables;[88] together with the angle variables $\boldsymbol{\varphi}$ they form the *action-angle system of canonical coordinates* in a neighborhood of $M_{\mathbf{f}}$.

The quantities I_i are first integrals of the system with hamiltonian function $H = F_1$, since they are functions of the first integrals F_j. In turn, the variables F_i can be expressed in terms of $\mathbf{I}$ and, in particular, $H = F_1 = H(\mathbf{I})$. In action-angle variables the differential equations of our flow (1) have the form

$$\frac{d\mathbf{I}}{dt} = 0 \qquad \frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}(\mathbf{I}). \tag{2}$$

PROBLEM. Can the functions $\boldsymbol{\omega}(\mathbf{I})$ in (2) be arbitrary?

Solution. In the variables $(\mathbf{I}, \boldsymbol{\varphi})$, the equations of the flow (2) have the canonical form with hamiltonian function $H(\mathbf{I})$. Therefore, $\boldsymbol{\omega}(\mathbf{I}) = \partial H/\partial \mathbf{I}$; thus if the number of degrees of freedom is $n \geq 2$, the functions $\boldsymbol{\omega}(\mathbf{I})$ are not arbitrary, but satisfy the symmetry condition $\partial\omega_i/\partial I_j = \partial\omega_j/\partial I_i$.

Action-angle variables are especially important for perturbation theory; in Section 52 we will demonstrate their application to the theory of adiabatic invariants.

B Construction of action-angle variables in the case of one degree of freedom

A system with one degree of freedom in the phase plane (p, q) is given by the hamiltonian function $H(p, q)$.

EXAMPLE 1. The harmonic oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$; or, more generally, $H = \frac{1}{2}a^2p^2 + \frac{1}{2}b^2q^2$.

EXAMPLE 2. The mathematical pendulum $H = \frac{1}{2}p^2 - \cos q$. In both cases we have a compact closed curve $M_h(H = h)$, and the conditions of the theorem of Section 49 for $n = 1$ are satisfied.

In order to construct the action-angle variables, we will look for a canonical transformation $(p, q) \to (I, \varphi)$ satisfying the two conditions:

$$\begin{aligned} &1.\ I = I(h), \\ &2.\ \oint_{M_h} d\varphi = 2\pi. \end{aligned} \tag{3}$$

PROBLEM. Find the action-angle variables in the case of the simple harmonic oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$.

Solution. If r, φ are polar coordinates, then $dp \wedge dq = r\,dr \wedge d\varphi = d(r^2/2) \wedge d\varphi$. Therefore, $I = H = (p^2 + q^2)/2$.

[88] It is not hard to see that $\mathbf{I}$ has the dimensions of action.

In order to construct the canonical transformation $p, q \to I, \varphi$ in the general case, we will look for its generating function $S(I, q)$:

$$(4)\qquad p = \frac{\partial S(I, q)}{\partial q} \qquad \varphi = \frac{\partial S(I, q)}{\partial I} \qquad H\left(\frac{\partial S(I, q)}{\partial q}, q\right) = h(I).$$

We first assume that the function $h(I)$ is known and invertible, so that every curve M_h is determined by the value of I ($M_h = M_{h(I)}$). Then for a fixed value of I we have from (4)

$$dS|_{I=\text{const}} = p\, dq.$$

This relation determines a well-defined differential 1-form dS on the curve $M_{h(I)}$.

Integrating this 1-form on the curve $M_{h(I)}$ we obtain (in a neighborhood of a point q_0) a function

$$S(I, q) = \int_{q_0}^{q} p\, dq.$$

This function will be the generating function of the transformation (4) in a neighborhood of the point (I, q_0). The first of the conditions (3) is satisfied automatically: $I = I(h)$. To verify the second condition, we consider the behavior of $S(I, q)$ "in the large." After a circuit of the closed curve $M_{h(I)}$ the integral of $p\, dq$ increases by

$$\Delta S(I) = \oint_{M_{h(I)}} p\, dq,$$

equal to the area Π enclosed by the curve $M_{h(I)}$. Therefore, the function S is a "multiple-valued function" on $M_{h(I)}$: it is determined up to addition of integral multiples of Π. This term has no effect on the derivative $\partial S(I, q)/\partial q$; but it leads to the multi-valuedness of $\varphi = \partial S/\partial I$. This derivative turns out to be defined only up to multiples of $d\, \Delta S(I)/dI$. More precisely, the formulas (4) define a 1-form $d\varphi$ on the curve $M_{h(I)}$, and the integral of this form on $M_{h(I)}$ is equal to $d\, \Delta S(I)/dI$.

In order to fulfill the second condition, $\oint_{M_h} d\varphi = 2\pi$, we need that

$$\frac{d}{dI}\Delta S(I) = 2\pi \qquad I = \frac{\Delta S}{2\pi} = \frac{\Pi}{2\pi},$$

where $\Pi = \oint_{M_h} p\, dq$ is the area bounded by the phase curve $H = h$.

Definition. The *action variable* in the one-dimensional problem with hamiltonian function $H(p, q)$ is the quantity $I(h) = (1/2\pi)\Pi(h)$.

Finally, we arrive at the following conclusion. Let $d\Pi/dh \neq 0$. Then the inverse $I(h)$ of the function $h(I)$ is defined.

Theorem. *Set* $S(I, q) = \int_{q_0}^{q} p\, dq|_{H=h(I)}$. *Then formulas* (4) *give a canonical transformation* $p, q \to I, \varphi$ *satisfying conditions* (3).

Thus, the action-angle variables in the one-dimensional case are constructed.

PROBLEM. Find S and I for a harmonic oscillator.

ANSWER. If $H = \frac{1}{2}a^2p^2 + \frac{1}{2}b^2q^2$ (Figure 217), then M_h is the ellipse bounding the area $\Pi(h) = \pi(\sqrt{2h}/a)(\sqrt{2h}/b) = 2\pi h/ab = 2\pi h/\omega$. Thus for a harmonic oscillator the action variable is the ratio of energy to frequency. The angle variable φ is, of course, the phase of oscillation.

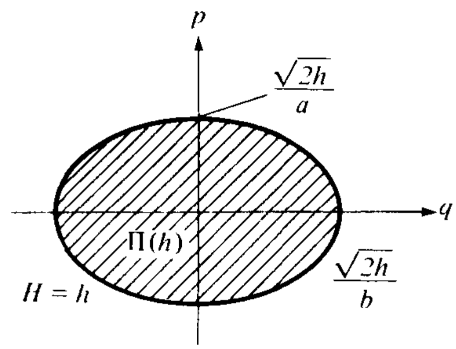

Figure 217 Action variable for a hamonic oscillator

PROBLEM. Show that the period T of motion along the closed curve $H = h$ on the phase plane p, q is equal to the derivative with respect to h of the area bounded by this curve:

$$T = \frac{d\Pi(h)}{dh}.$$

Solution. In action-angle variables the equations of motion (2) give

$$\dot{\varphi} = \frac{\partial H}{\partial I} = \left(\frac{dI}{\partial h}\right)^{-1} = 2\pi\left(\frac{d\Pi}{dh}\right)^{-1} \qquad T = \frac{2\pi}{\dot{\varphi}} = \frac{d\Pi}{dh}.$$

C *Construction of action-angle variables in* $\mathbb{R}^{2n}$

We turn now to systems with n degrees of freedom given in $\mathbb{R}^{2n} = \{(\mathbf{p}, \mathbf{q})\}$ by a hamiltonian function $H(\mathbf{p}, \mathbf{q})$ and having n first integrals in involution $F_1 = H, F_2, \ldots, F_n$. We will not repeat the reasoning which brought us to the choice of $2\pi I = \oint p\, dq$ in the one-dimensional case, but will immediately define n action variables $\mathbf{I}$.

Let $\gamma_1, \ldots, \gamma_n$ be a basis for the one-dimensional cycles on the torus $M_\mathbf{f}$ (the increase of the coordinate φ_i on the cycle γ_j is equal to 2π if $i = j$ and 0 if $i \neq j$). We set

$$(5) \qquad I_i(\mathbf{f}) = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p}\, d\mathbf{q}.$$

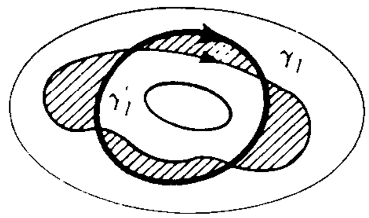

Figure 218 Independence of the curve of integration for the action variable

PROBLEM. Show that this integral does not depend on the choice of the curve γ_i representing the cycle (Figure 218).

Hint. In Section 49 we showed that the 2-form $\omega^2 = \sum dp_i \wedge dq_i$ on the manifold $M_{\mathbf{f}}$ is equal to zero. By Stokes' formula,

$$\oint_{\gamma_i} - \oint_{\gamma'_i} \mathbf{p}\, d\mathbf{q} = \iint_\sigma d\mathbf{p} \wedge d\mathbf{q} = 0,$$

where $\partial\sigma = \gamma - \gamma'$.

Definition. The n quantities $I_i(\mathbf{f})$ given by formula (5) are called the *action variables.*

We assume now that, for the given values f_i of the n integrals F_i, the n quantities I_i are independent: $\det(\partial \mathbf{I}/\partial \mathbf{f})|_{\mathbf{f}} \neq 0$. Then in a neighborhood of the torus $M_{\mathbf{f}}$ we can take the variables $\mathbf{I}$, $\boldsymbol{\varphi}$ as coordinates.

Theorem. *The transformation* $\mathbf{p}, \mathbf{q} \to \mathbf{I}, \boldsymbol{\varphi}$ *is canonical, i.e.,*

$$\sum dp_i \wedge dq_i = \sum dI_i \wedge d\varphi_i.$$

We outline the proof of this theorem. Consider the differential 1-form $\mathbf{p}\, d\mathbf{q}$ on $M_{\mathbf{f}}$. Since the manifold $M_{\mathbf{f}}$ is null (Section 49) this 1-form on $M_{\mathbf{f}}$ is closed: its exterior derivative $\omega^2 = d\mathbf{p} \wedge d\mathbf{q}$ is identically equal to zero on $M_{\mathbf{f}}$. Therefore (Figure 219),

$$S(x) = \int_{x_0}^{x} \mathbf{p}\, d\mathbf{q}|_{M_{\mathbf{f}}}$$

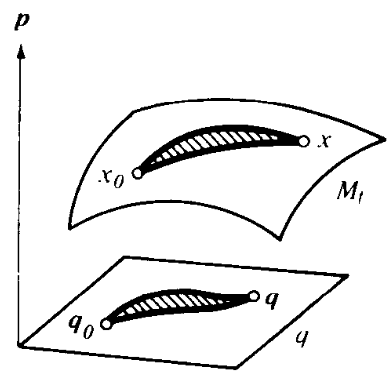

Figure 219 Independence of the path for the integral of $\mathbf{p}\, d\mathbf{q}$ on M_f

does not change under deformations of the path of integration (Stokes' formula). Thus $S(x)$ is a "multiple-valued function" on $M_{\mathbf{f}}$, with periods equal to

$$\Delta_i S = \int_{\gamma_i} dS = 2\pi I_i.$$

Now let x_0 be a point on $M_{\mathbf{f}}$, in a neighborhood of which the n variables $\mathbf{q}$ are coordinates on $M_{\mathbf{f}}$, such that the submanifold $M_{\mathbf{f}} \subset \mathbb{R}^{2n}$ is given by n equations of the form $\mathbf{p} = \mathbf{p}(\mathbf{I}, \mathbf{q}), \mathbf{q}(x_0) = \mathbf{q}_0$. In a simply connected neighborhood of the point $\mathbf{q}_0$ a single-valued function is defined,

$$S(\mathbf{I}, \mathbf{q}) = \int_{q_0}^{q} \mathbf{p}(\mathbf{I}, \mathbf{q}) d\mathbf{q},$$

and we can use it as the generating function of a canonical transformation $\mathbf{p}, \mathbf{q} \to \mathbf{I}, \boldsymbol{\varphi}$:

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} \qquad \boldsymbol{\varphi} = \frac{\partial S}{\partial \mathbf{I}}.$$

It is not difficult to verify that these formulas actually give a canonical transformation, not only in a neighborhood of the point under consideration, but also "in the large" in a neighborhood of $M_{\mathbf{f}}$. The coordinates $\boldsymbol{\varphi}$ will be multiple-valued with periods

$$\Delta_i \varphi_j = \Delta_i \frac{\partial S}{\partial I_j} = \frac{\partial}{\partial I_j} \Delta_i S = \frac{\partial}{\partial I_j} 2\pi I_i = 2\pi \delta_{ij},$$

as was to be shown. □

We now note that all our constructions involve only "algebraic" operations (inverting functions) and "quadrature"—calculation of the integrals of known functions. In this way the problem of integrating a canonical system with $2n$ equations, of which n first integrals in involution are known, is solved by quadratures, which proves the last assertion of Liouville's theorem (Section 49). □

Remark 1. Even in the one-dimensional case the action-angle variables are not uniquely defined by the conditions (3). We could have taken $I' = I + \text{const}$ for the action variable and $\varphi' = \varphi + c(I)$ for the angle variable.

Remark 2. We constructed action-angle variables for systems with phase space $\mathbb{R}^{2n}$. We could also have introduced action-angle variables for a system on an arbitrary symplectic manifold. We restrict outselves here to one simple example (Figure 220).

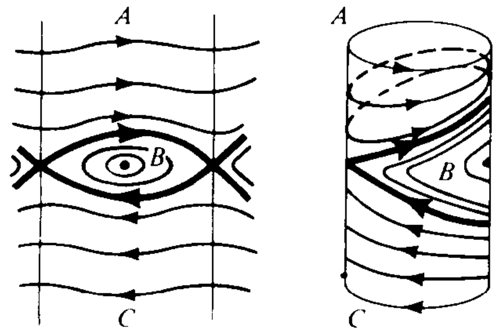

Figure 220 Action-angle variables on a symplectic manifold

We could have taken the phase space of a pendulum ($H = \frac{1}{2}p^2 - \cos q$) to be, instead of the plane $\{(p, q)\}$, the surface of the cylinder $\mathbb{R}^1 \times S^1$ obtained by identifying angles q differing by an integral multiple of 2π.

The critical level lines $H = \pm 1$ divide the cylinder into three parts, A, B, and C, each of which is diffeomorphic to the direct product $\mathbb{R}^1 \times S^1$. We can introduce action-angle variables into each part. In the bounded part (B) the closed trajectories represent the oscillation of the pendulum; in the unbounded parts they represent rotation.

Remark 3. In the general case, as in the example analyzed above, the equations $F_i = f_i$ cease to be independent for some values of f_i, and $M_{\mathbf{f}}$ ceases to be a manifold. Such critical values of $\mathbf{f}$ correspond to separatrices dividing the phase space of the integrable problem into parts corresponding to the parts A, B, and C above. In some of these parts the manifolds $M_{\mathbf{f}}$ can be unbounded (parts A and C in the plane $\{(p, q)\}$); others are stratified into n-dimensional invariant tori $M_{\mathbf{f}}$; in a neighborhood of such a torus we can introduce action-angle variables.

51 Averaging

In this paragraph we show that time averages and space averages are equal for systems undergoing conditionally-periodic motion.

A *Conditionally-periodic motion*

In the earlier sections of this book. we have frequently encountered conditionally-periodic motion: Lissajous figures, precession, nutation, rotation of a top, etc.

Definition. Let T^n be the n-dimensional torus and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n) \bmod 2\pi$ angular coordinates. Then by a *conditionally-periodic motion* we mean a one-parameter group of diffeomorphisms $T^n \to T^n$ given by the differential equations (Figure 221):

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) = \textbf{const}.$$

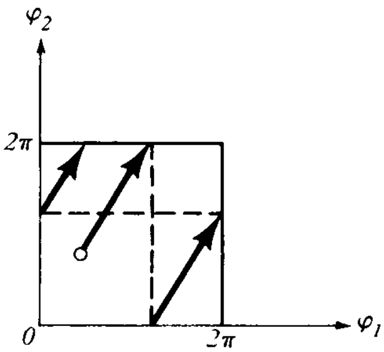

Figure 221 Conditionally-periodic motion

These differential equations are easily integrated:

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega} t.$$

Thus the trajectories in the chart $\{\boldsymbol{\varphi}\}$ are straight lines. A trajectory on the torus is called a *winding* of the torus.

EXAMPLE. Let $n = 2$. If $\omega_1/\omega_2 = k_1/k_2$, the trajectories are closed; if ω_1/ω_2 is irrational, then trajectories on the torus are dense (cf. Section 16).

The quantities $\omega_1, \ldots, \omega_n$ are called the *frequencies* of the conditionally-periodic motion. The frequencies are called *independent* if they are linearly independent over the field of rational numbers: if $\mathbf{k} \in \mathbb{Z}^n$[89] and $(\mathbf{k}, \boldsymbol{\omega}) = 0$, then $\mathbf{k} = 0$.

B *Space average and time average*

Let $f(\boldsymbol{\varphi})$ be an integrable function on the torus T^n.

Definition. The *space* average of a function f on the torus T^n is the number

$$\bar{f} = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\boldsymbol{\varphi}) d\varphi_1 \cdots d\varphi_n.$$

Consider the value of the function $f(\boldsymbol{\varphi})$ on the trajectory $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_0 + \boldsymbol{\omega} t$. This is a function of time, $f(\boldsymbol{\varphi}_0 + \boldsymbol{\omega} t)$. We consider its average.

Definition. The *time average* of the function f on the torus T^n is the function

$$f^*(\boldsymbol{\varphi}_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\boldsymbol{\varphi}_0 + \boldsymbol{\omega} t) dt$$

(defined where the limit exists).

Theorem on the averages. *The time average exists everywhere, and coincides with the space average if f is continuous (or merely Riemann integrable) and the frequencies ω_i are independent.*

[89] $\mathbf{k} = (k_1, \ldots, k_n)$ with integral k_i.

PROBLEM. Show that if the frequencies are dependent, then the time average can differ from the space average.

Corollary 1. *If the frequencies are independent, then every trajectory* $\{\boldsymbol{\varphi}(t)\}$ *is dense on the torus* T^n.

PROOF. Assume the contrary. Then in some neighborhood D of some point of the torus, there is no point of the trajectory $\boldsymbol{\varphi}(t)$. It is easy to construct a continuous function f equal to zero outside D and with space average equal to 1. The time average $f^*(\boldsymbol{\varphi}_0)$ on the trajectory $\boldsymbol{\varphi}(t)$ is equal to $0 \neq 1$. This contradicts the assertion of the theorem. □

Corollary 2. *If the frequencies are independent, then every trajectory is uniformly distributed on the torus* T^n.

This means that the time the trajectory spends in a neighborhood D is proportional to the measure of D.

More precisely, let D be a (Jordan) measurable region of T^n. We denote by $\tau_D(T)$ the amount of time that the interval $0 \leq t \leq T$ of the trajectory $\boldsymbol{\varphi}(t)$ is inside of D. Then

$$\lim_{T \to \infty} \frac{\tau_D(T)}{T} = \frac{\text{mes } D}{(2\pi)^n}.$$

PROOF. We apply the theorem to the characteristic function f of the set D (f is Riemann integrable since D is Jordan measurable). Then $\int_0^T f(\boldsymbol{\varphi}(t))dt = \tau_D(T)$, and $\bar{f} = (2\pi)^{-n} \text{ mes } D$, and the corollary follows immediately from the theorem. □

Corollary. *In the sequence*

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, \ldots$$

of first digits of the numbers 2^n, *the number* 7 *appears* (log 8 − log 7)/(log 9 − log 8) *times as often as* 8.

The theorem on averages may be found implicitly in the work of Laplace, Lagrange, and Gauss on celestial mechanics; it is one of the first "ergodic theorems." A rigorous proof was given only in 1909 by P. Bohl, W. Sierpinski, and H. Weyl in connection with a problem of Lagrange on the mean motion of the earth's perihelion. Below we reproduce H. Weyl's proof.

C *Proof of the theorem on averages*

Lemma 1. *The theorem is true for exponentials* $f = e^{i(\mathbf{k}, \boldsymbol{\varphi})}$, $\mathbf{k} \in \mathbb{Z}^n$.

PROOF. If $\mathbf{k} = 0$, then $\bar{f} = f = f^* = 1$ and the theorem is obvious. If $\mathbf{k} \neq 0$, then $\bar{f} = 0$. On the other hand,

$$\int_0^T e^{i(\mathbf{k}, \boldsymbol{\varphi}_0 + \boldsymbol{\omega} t)}\, dt = e^{i(\mathbf{k}, \boldsymbol{\varphi}_0)} \frac{e^{i(\mathbf{k}, \boldsymbol{\omega})T} - 1}{i(\mathbf{k}, \boldsymbol{\omega})}.$$

Therefore, the time average is

$$\lim_{T\to\infty} \frac{e^{i(\mathbf{k},\boldsymbol{\varphi}_0)}}{i(\mathbf{k},\boldsymbol{\omega})} \frac{e^{i(\mathbf{k},\boldsymbol{\omega})T}-1}{T} = 0. \qquad \square$$

Lemma 2. *The theorem is true for trigonometric polynomials*

$$f = \sum_{|\mathbf{k}|<N} f_{\mathbf{k}} e^{i(\mathbf{k},\boldsymbol{\varphi})}.$$

PROOF. Both the time and space averages depend linearly on f, and therefore agree by Lemma 1. □

Lemma 3. *Let f be a real continuous (or at least Riemann integrable) function. Then, for any $\varepsilon > 0$, there exist two trigonometric polynomials P_1 and P_2 such that $P_1 < f < P_2$ and $(1/(2\pi)^n)\int_{T^n}(P_2 - P_1)d\boldsymbol{\varphi} \le \varepsilon$.*

PROOF. Suppose first that f is continuous. By the Weierstrass theorem, we can approximate f by a trigonometric polynomial P with $|f - P| < \frac{1}{2}\varepsilon$. The polynomials $P_1 = P - \frac{1}{2}\varepsilon$ and $P_2 = P + \frac{1}{2}\varepsilon$ are the ones we are looking for.

If f is not continuous but Riemann integrable, then there are two continuous functions f_1 and f_2 such that $f_1 < f < f_2$ and $(2\pi)^{-n}\int(f_2 - f_1)d\boldsymbol{\varphi} < \frac{1}{3}\varepsilon$ (Figure 222 corresponds to the characteristic function of an interval). By approximating f_1 and f_2 by polynomials $P_1 < f_1 < f_2 < P_2$, $(2\pi)^{-n}\int(P_2 - f_2)d\boldsymbol{\varphi} < \frac{1}{3}\varepsilon$, $(2\pi)^{-n}\int(f_1 - P_1)d\boldsymbol{\varphi} < \frac{1}{3}\varepsilon$, we obtain what we need. Lemma 3 is proved. □

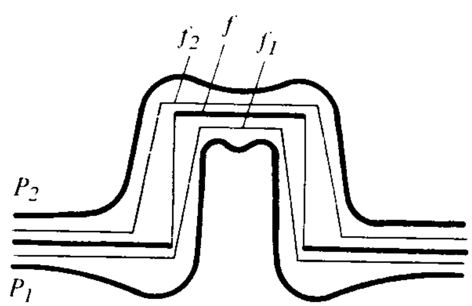

Figure 222 Approximation of the function f by trigonometric polynomials P_1 and P_2

It is now easy to finish the proof of the theorem. Let $\varepsilon > 0$. Then, by Lemma 3, there are trigonometric polynomials $P_1 < f < P_2$ with $(2\pi)^{-n}\int(P_2 - P_1)d\boldsymbol{\varphi} < \varepsilon$.

For any T, we then have

$$\frac{1}{T}\int_0^T P_1(\boldsymbol{\varphi}(t))dt < \frac{1}{T}\int_0^T f(\boldsymbol{\varphi}(t))dt < \frac{1}{T}\int_0^T P_2(\boldsymbol{\varphi}(t))dt.$$

By Lemma 2, for $T > T_0(\varepsilon)$,

$$\left|\bar{P}_i - \frac{1}{T}\int_0^T P_i(\boldsymbol{\varphi}(t))dt\right| < \varepsilon \qquad (i = 1, 2).$$

Furthermore, $\bar{P}_1 < \bar{f} < \bar{P}_2$ and $\bar{P}_2 - \bar{P}_1 < \varepsilon$. Therefore, $\bar{P}_2 - \bar{f} < \varepsilon$ and $\bar{f} - \bar{P}_1 < \varepsilon$; therefore, for $T > T_0(\varepsilon)$,

$$\left| \frac{1}{T} \int_0^T f(\boldsymbol{\varphi}(t))dt - \bar{f} \right| < 2\varepsilon,$$

as was to be proved. □

PROBLEM. A two-dimensional oscillator with kinetic energy $T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2$ and potential energy $U = \frac{1}{2}x^2 + y^2$ performs an oscillation with amplitudes $a_x = 1$ and $a_y = 1$. Find the time average of the kinetic energy.

PROBLEM.[90] Let ω_k be independent, $a_k > 0$. Calculate

$$\lim_{t\to\infty} \frac{1}{t} \arg \sum_{k=1}^{3} a_k e^{i\omega_k t}.$$

ANSWER. $(\omega_1\alpha_1 + \omega_2\alpha_2 + \omega_3\alpha_3)/\pi$, where α_1, α_2, and α_3 are the angles of the triangle with sides a_k (Figure 223).

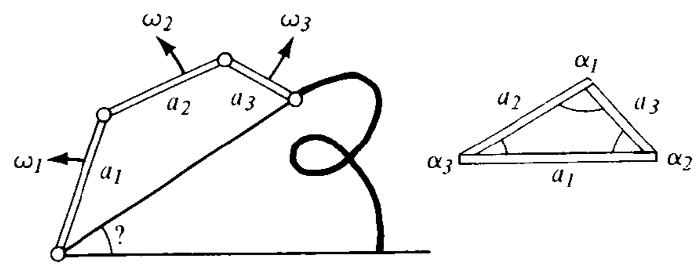

Figure 223 Problem on mean motion of perihelia

D *Degeneracies*

So far we have considered the case when the frequencies $\boldsymbol{\omega}$ are independent. An integral vector $\mathbf{k} \in \mathbb{Z}^n$ is called a *relation among the frequencies* if $(\mathbf{k}, \boldsymbol{\omega}) = 0$.

PROBLEM. Show that the set of all relations between a given set of frequencies $\boldsymbol{\omega}$ is a subgroup Γ of the lattice $\mathbb{Z}^n$.

We saw in Section 49 that such a subgroup consists entirely of linear combinations of r independent vectors $\mathbf{k}_i$, $1 \le r \le n$. We say that there are r *(independent) relations* among the frequencies.[91]

[90] Lagrange showed that the investigation of the average motion of the perihelion of a planet reduces to a similar problem. The solution of this problem can be found in the work of H. Weyl. The eccentricity of the earth's orbit varies as the modulus of an analogous sum. Ice ages appear to be related to these changes in eccentricity.

[91] Show that the number r does not depend on the choice of independent vectors $\mathbf{k}_i$.

PROBLEM. Show that the closure of a trajectory $\{\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_0 + \boldsymbol{\omega}t\}$ (on T^n) is a torus of dimension $n - r$ if there are r independent relations among the frequencies $\boldsymbol{\omega}$; in this case the motion on T^{n-r} is conditionally-periodic with $n - r$ independent frequencies.

We turn now to the integrable hamiltonian system given in action-angle variables $\mathbf{I}, \boldsymbol{\varphi}$ by the equations

$$\dot{\mathbf{I}} = 0 \qquad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(\mathbf{I}), \quad \text{where } \boldsymbol{\omega}(\mathbf{I}) = \frac{\partial H}{\partial \mathbf{I}}.$$

Every n-dimensional torus $\mathbf{I} = \mathbf{const}$ in the $2n$-dimensional phase space is invariant, and motion on it is conditionally-periodic.

Definition. A system is called *nondegenerate* if the determinant

$$\det \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{I}} = \det \frac{\partial^2 H}{\partial \mathbf{I}^2}$$

is not zero.

PROBLEM. Show that, if a system is nondegenerate, then in any neighborhood of any point there is a conditionally-periodic motion with n frequencies, and also with any smaller number of frequencies.

Hint. We can take the frequencies $\boldsymbol{\omega}$ themselves instead of the variables $\mathbf{I}$ as local coordinates. In the space of collections of frequencies, the set of points $\boldsymbol{\omega}$ with any number of relations $r (0 \leq r < n)$ is dense.

Corollary. *If a system is nondegenerate, then the invariant tori* $\mathbf{I} = \mathbf{const}$ *are uniquely defined, independent of the choice of action-angle coordinates* $\mathbf{I}, \boldsymbol{\varphi}$, *the construction of which always involves some arbitrariness.*[92]

PROOF. The tori $\mathbf{I} = \mathbf{const}$ can be defined as the closures of the phase trajectories corresponding to the independent $\boldsymbol{\omega}$. □

We note incidentally that, for the majority of values $\mathbf{I}$, the frequencies $\boldsymbol{\omega}$ will be independent.

PROBLEM. Show that the set of $\mathbf{I}$ for which the frequencies $\boldsymbol{\omega}(\mathbf{I})$ in a nondegenerate system are dependent has Lebesgue measure equal to zero.

Hint. Show first that

$$\operatorname{mes} \{\boldsymbol{\omega} : \exists \mathbf{k} \neq 0, (\boldsymbol{\omega}, \mathbf{k}) = 0\} = 0.$$

On the other hand, in degenerate systems we can construct systems of action-angle variables such that the tori $\mathbf{I} = \mathbf{const}$ will be different in different systems. This is the case because the closures of trajectories in a degenerate system are tori of dimension $k < n$, and they can be contained in different ways in n-dimensional tori.

[92] For example, we can always write the substitution $\mathbf{I}' = \mathbf{I}$, $\boldsymbol{\varphi}' = \boldsymbol{\varphi} + S_1(\mathbf{I})$, or $I_1, I_2: \varphi_1, \varphi_2 \to I_1 + I_2, I_2: \varphi_1, \varphi_2 - \varphi_1$.

EXAMPLE 1. The planar harmonic oscillator $\ddot{\mathbf{x}} = -\mathbf{x}$; $n = 2$, $k = 1$. Separation of variables in cartesian and polar coordinates leads to different action-angle variables and different tori.

EXAMPLE 2. Keplerian planar motion ($U = -1/r$), $n = 2$, $k = 1$. Here, too, separation of variables in polar and in elliptical coordinates leads to different $\mathbf{I}$.

52 Averaging of perturbations

Here we show the adiabatic invariance of the action variable in a system with one degree of freedom.

A *Systems close to integrable ones*

We have considered a great many integrable systems (one-dimensional problems, the two-body problem, small oscillations, the Euler and Lagrange cases of the motion of a rigid body with a fixed point, etc.). We studied the characteristics of phase trajectories in these systems: they turned out to be "windings of tori," densely filling up the invariant tori in phase space; every trajectory is uniformly distributed on this torus.

One should not conclude from this that integrability is the typical situation. Actually, the properties of trajectories in many-dimensional systems can be highly diverse and not at all similar to the properties of conditionally-periodic motions. In particular, the closure of a trajectory of a system with n degrees of freedom can fill up complicated sets of dimension greater than n in $2n$-dimensional phase space; a trajectory could even be dense and uniformly distributed on a whole $(2n - 1)$-dimensional manifold given by the equation $H = h$.[93] One may call such systems "nonintegrable" since they do not admit single-valued first integrals independent of H. The study of such systems is still far from complete; it constitutes a problem in "ergodic theory."

One approach to nonintegrable systems is to study systems which are close to integrable ones. For example, the problem of the motion of planets around the sun is close to the integrable problem of the motion of non-interacting points around a stationary center; other examples are the problem of the motion of a slightly nonsymmetric heavy top and the problem of nonlinear oscillations close to an equilibrium position (the nearby integrable problem is linear). The following method is especially fruitful in the investigation of these and similar problems.

B *The averaging principle*

Let $\mathbf{I}$, $\boldsymbol{\varphi}$ be action-angle variables in an integrable ("nonperturbed") system with hamiltonian function $H_0(\mathbf{I})$:

$$\dot{\mathbf{I}} = 0 \qquad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(\mathbf{I}) \qquad \boldsymbol{\omega}(\mathbf{I}) = \frac{\partial H_0}{\partial \mathbf{I}}.$$

[93] For example, inertial motion on a manifold of negative curvature has this property.

As the nearby "perturbed" system we take the system

$$(1) \qquad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(\mathbf{I}) + \varepsilon \mathbf{f}(\mathbf{I}, \boldsymbol{\varphi}) \qquad \dot{\mathbf{I}} = \varepsilon \mathbf{g}(\mathbf{I}, \boldsymbol{\varphi}),$$

where $\varepsilon \ll 1$.

We will ignore for a while that the system is hamiltonian and consider an arbitrary system of differential equations in the form (1) given on the direct product $T^k \times G$ of the k-dimensional torus $T^k = \{\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k) \bmod 2\pi\}$ and a region G in l-dimensional space $G \subset \mathbb{R}^l = \{\mathbf{I} = (I_1, \ldots, I_l)\}$. For $\varepsilon = 0$ the motion in (1) is conditionally-periodic with at most k frequencies and with k-dimensional invariant tori.

The *averaging principle for system* (1) consists of its replacement by another system, called the averaged system:

$$(2) \qquad \dot{\mathbf{J}} = \varepsilon \bar{\mathbf{g}}(\mathbf{J}) \qquad \bar{\mathbf{g}}(\mathbf{J}) = (2\pi)^{-k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbf{g}(\mathbf{J}, \boldsymbol{\varphi}) d\varphi_1, \ldots, d\varphi_k$$

in the l-dimensional region $G \subset \mathbb{R}^l = \{\mathbf{J} = (J_1, \ldots, J_l)\}$.

We claim that system (2) is a "good approximation" to system (1).

We note that this principle is neither a theorem, an axiom, nor a definition, but rather a physical proposition, i.e., a vaguely formulated and, strictly speaking, untrue assertion. Such assertions are often fruitful sources of mathematical theorems.

This averaging principle may be found explicitly in the work of Gauss (in studying the perturbations of planets on one another, Gauss proposed to distribute the mass of each planet around its orbit proportionally to time and to replace the attraction of each planet by the attraction of the ring so obtained). Nevertheless, a satisfactory description of the connection between the solutions of systems (1) and (2) in the general case has not yet been found.

In replacing system (1) by system (2) we discard the term $\varepsilon \tilde{\mathbf{g}}(\mathbf{I}, \boldsymbol{\varphi}) = \varepsilon \mathbf{g}(\mathbf{I}, \boldsymbol{\varphi}) - \varepsilon \bar{\mathbf{g}}(\mathbf{I})$ on the right-hand side. This term has order ε as does the remaining term $\varepsilon \bar{\mathbf{g}}$. In order to understand the different roles of the terms $\bar{\mathbf{g}}$ and $\tilde{\mathbf{g}}$ in $\mathbf{g}$, we consider the simplest example.

PROBLEM. Consider the case $k = l = 1$,

$$\dot{\varphi} = \omega \neq 0 \qquad \dot{I} = \varepsilon g(\varphi).$$

Show that for $0 < t < 1/\varepsilon$,

$$|I(t) - J(t)| < c\varepsilon, \quad \text{where } J(t) = I(0) + \varepsilon \bar{g} t.$$

Solution

$$I(t) - I(0) = \int_0^t \varepsilon g(\varphi_0 + \omega t) dt = \int_0^t \varepsilon \bar{g}\, dt + \frac{\varepsilon}{\omega} \int_0^{\omega t} \tilde{g}(\varphi) d\varphi = \varepsilon \bar{g} t + \frac{\varepsilon}{\omega} h(\omega t)$$

where $h(\varphi) = \int_0^\varphi \tilde{g}(\varphi) d\varphi$ is a periodic, and therefore bounded, function.

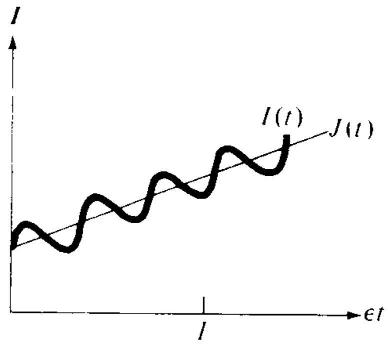

Figure 224 Evolution and oscillation

Thus the variation in I with time consists of two parts: an oscillation of order ε depending on $\tilde{g}$ and a systematic "evolution" with velocity $\varepsilon\bar{g}$ (Figure 224).

The averaging principle is based on the assertion that in the general case the motion of system (1) can be divided into the "evolution" (2) and small oscillations. In its general form, this assertion is invalid and the principle itself is untrue. Nevertheless, we will apply the principle to the hamiltonian system (1):

$$\dot{\boldsymbol{\varphi}} = -\frac{\partial}{\partial \mathbf{I}}(H_0(\mathbf{I}) + \varepsilon H_1(\mathbf{I}, \boldsymbol{\varphi})) \qquad \dot{\mathbf{I}} = \frac{\partial}{\partial \boldsymbol{\varphi}}(H_0(\mathbf{I}) + \varepsilon H_1(\mathbf{I}, \boldsymbol{\varphi})).$$

For the right-hand side of the averaged system (2) we then obtain

$$\bar{\mathbf{g}} = (2\pi)^{-n}\int_0^{2\pi} \frac{\partial}{\partial \boldsymbol{\varphi}} H_1(\mathbf{I}, \boldsymbol{\varphi})d\boldsymbol{\varphi} = 0.$$

In other words, there is no evolution in a nondegenerate hamiltonian system.

One variant of this entirely nonrigorous deduction leads to the so-called Laplace theorem: The semi-major axes of the keplerian ellipses of the planets have no secular perturbations.

The discussion above suffices to convince us of the importance of the averaging principle; we now formulate a theorem justifying this principle in one very particular case—that of single-frequency oscillations ($k = 1$). This theorem shows that the averaging principle correctly describes evolution over a large interval of time ($0 < t < 1/\varepsilon$).

C *Averaging in a single-frequency system*

Consider the system of $l + 1$ differential equations

$$(1) \qquad \left.\begin{aligned} \dot{\varphi} &= \omega(\mathbf{I}) + \varepsilon f(\mathbf{I}, \varphi) \\ \dot{\mathbf{I}} &= \varepsilon \mathbf{g}(\mathbf{I}, \varphi) \end{aligned}\right\} \qquad \begin{aligned} &\varphi \bmod 2\pi \in S^1, \\ &\mathbf{I} \in G \subset \mathbb{R}^l, \end{aligned}$$

where $f(\mathbf{I}, \varphi + 2\pi) \equiv f(\mathbf{I}, \varphi)$ and $\mathbf{g}(\mathbf{I}, \varphi + 2\pi) \equiv \mathbf{g}(\mathbf{I}, \varphi)$, together with the "averaged" system of l equations

$$(2) \qquad \dot{\mathbf{J}} = \varepsilon\bar{\mathbf{g}}(\mathbf{J}), \quad \text{where } \bar{\mathbf{g}}(\mathbf{J}) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{g}(\mathbf{J}, \varphi)d\varphi.$$

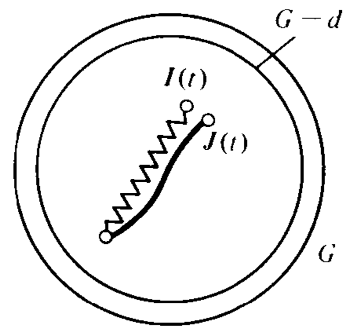

Figure 225 Theorem on averaging

We denote by $\mathbf{I}(t)$, $\varphi(t)$ the solution of system (1) with initial conditions $\mathbf{I}(0)$, $\varphi(0)$, and by $\mathbf{J}(t)$ the solution of system (2) with the same initial conditions $\mathbf{J}(0) = \mathbf{I}(0)$ (Figure 225).

Theorem. *Suppose that:*

1. *the functions ω, f, and $\mathbf{g}$ are defined for $\mathbf{I}$ in a bounded region G, and in this region they are bounded, together with their derivatives up to second order:*

$$\|\omega, f, \mathbf{g}\|_{C^2(G \times S^1)} < c_1;$$

2. *in the region G, we have*

$$\omega(\mathbf{I}) > c > 0;$$

3. *for $0 \le t \le 1/\varepsilon$, a neighborhood of radius d of the point $\mathbf{J}(t)$ belongs to G:*

$$\mathbf{J}(t) \in G - d.$$

Then for sufficiently small ε $(0 < \varepsilon < \varepsilon_0)$

$$|\mathbf{I}(t) - \mathbf{J}(t)| < c_9 \varepsilon, \quad \textit{for all } t, 0 \le t \le \frac{1}{\varepsilon},$$

where the constant $c_9 > 0$ depends on c_1, c, and d, but not on ε.

Some applications of this theorem will be given below ("adiabatic invariants"). We remark that the basic idea of the proof of this theorem (a change of variables diminishing the perturbation) is more important than the theorem itself; this is one of the basic ideas in the theory of ordinary differential equations; it is encountered in elementary courses as the "method of variation of constants."

D *Proof of the theorem on averaging*

In place of the variables $\mathbf{I}$ we will introduce new variables $\mathbf{P}$

$$\mathbf{P} = \mathbf{I} + \varepsilon \mathbf{k}(\mathbf{I}, \varphi), \tag{3}$$

where the function $\mathbf{k}$, 2π-periodic in φ, will be chosen so that the vector $\mathbf{P}$ will satisfy a simpler differential equation.

By (1) and (3), the rate of change of $\mathbf{P}(t)$ is

$$(4)\qquad \dot{\mathbf{P}} = \dot{\mathbf{I}} + \varepsilon\frac{\partial \mathbf{k}}{\partial \mathbf{I}}\dot{\mathbf{I}} + \varepsilon\frac{\partial \mathbf{k}}{\partial \varphi}\dot{\varphi} = \varepsilon\left[\mathbf{g}(\mathbf{I},\varphi) + \frac{\partial \mathbf{k}}{\partial \varphi}\omega(\mathbf{I})\right] + \varepsilon^2\frac{\partial \mathbf{k}}{\partial \mathbf{I}}\mathbf{g} + \varepsilon^2\frac{\partial \mathbf{k}}{\partial \varphi}f.$$

We assume that the substitution (3) can be inverted, so that

$$(5)\qquad \mathbf{I} = \mathbf{P} + \varepsilon\mathbf{h}(\mathbf{P},\varphi,\varepsilon)$$

(where the functions $\mathbf{h}$ are 2π-periodic in φ).

Then (4) and (5) imply that $\mathbf{P}(t)$ satisfies the system of equations

$$(6)\qquad \dot{\mathbf{P}} = \varepsilon\left[\mathbf{g}(\mathbf{P},\varphi) + \frac{\partial \mathbf{k}}{\partial \varphi}\omega(\mathbf{P})\right] + \mathbf{R},$$

where the "remainder term" $\mathbf{R}$ is small of second order with respect to ε:

$$(7)\qquad |\mathbf{R}| < c_2\varepsilon^2, \qquad c_2(c_1, c_3, c_4) > 0,$$

if only

$$(8)\qquad \|\omega\|_{C^2} < c_1 \quad \|f\|_{C^2} < c_1 \quad \|\mathbf{g}\|_{C^2} < c_1 \quad \|\mathbf{k}\|_{C^2} < c_3 \quad \|\mathbf{h}\|_{C^2} < c_4.$$

We will now try to choose the change of variables (3) so that the term involving ε in (6) becomes zero. For $\mathbf{k}$ we get the equation

$$\frac{\partial \mathbf{k}}{\partial \varphi} = -\frac{1}{\omega}\mathbf{g}.$$

In general, such an equation is not solvable in the class of functions $\mathbf{k}$ periodic in φ. In fact, the average value (with respect to φ) of the left-hand side is always equal to 0, and the average value of the right-hand side can be different from 0. Therefore, we cannot choose $\mathbf{k}$ in such a way as to kill the entire term involving ε in (6). However, we can kill the entire "periodic" part of $\mathbf{g}$,

$$\tilde{\mathbf{g}}(\mathbf{P},\varphi) = \mathbf{g}(\mathbf{P},\varphi) - \bar{\mathbf{g}}(\mathbf{P}),$$

by setting

$$(9)\qquad \mathbf{k}(\mathbf{P},\varphi) = -\int_0^{\varphi}\frac{\tilde{\mathbf{g}}(\mathbf{P},\varphi)}{\omega(\mathbf{P})}\,d\varphi.$$

So we define the function $\mathbf{k}$ by formula (9). Then, by hypotheses 1. and 2. of the theorem, the function $\mathbf{k}$ satisfies the estimate $\|\mathbf{k}\|_{C^2} < c_3$, where $c_3(c_1, c) > 0$. In order to establish the inequality (8), we must estimate $\mathbf{h}$. For this we must first show that the substitution (3) is invertible.

Fix a positive number α.

Lemma. *If ε is sufficiently small, then the restriction of the mapping* (3)[94]

$$\mathbf{I} \to \mathbf{I} + \varepsilon\mathbf{k}, \quad \textit{where } |\mathbf{k}|_{C^2(G)} < c_3,$$

[94] For any fixed value of the parameter φ.

to the region $G - \alpha$ (consisting of points whose α-neighborhood is contained in G) is a diffeomorphism. The inverse diffeomorphism (5) in the region $G - 2\alpha$ satisfies the estimate $\|\mathbf{h}\|_{C^2} < c_4$ with some constant $c_4(\alpha, c_3) > 0$.

PROOF. The necessary estimate follows directly from the implicit function theorem. The only difficulty is in verifying that the map $\mathbf{I} \to \mathbf{I} + \varepsilon\mathbf{k}$ is one-to-one in the region $G - \alpha$. We note that the function $\mathbf{k}$ satisfies a Lipschitz condition (with some constant $L(\alpha, c_3)$) in $G - \alpha$. Consider two points $\mathbf{I}_1$, $\mathbf{I}_2$ in $G - \alpha$. For sufficiently small ε (namely, for $L\varepsilon < 1$) the distance between $\varepsilon\mathbf{k}(\mathbf{I}_1)$ and $\varepsilon\mathbf{k}(\mathbf{I}_2)$ will be smaller than $|\mathbf{I}_1 - \mathbf{I}_2|$. Therefore, $\mathbf{I}_1 + \varepsilon\mathbf{k}(\mathbf{I}_1) \neq \mathbf{I}_2 + \varepsilon\mathbf{k}(\mathbf{I}_2)$. Thus the map (3) is one-to-one on $G - \alpha$, and the lemma is proved. □

It follows from the lemma that for ε small enough all the estimates (8) are satisfied. Thus the estimate (7) is also true.

We now compare the system of differential equations for $\mathbf{J}$

(2) $$\dot{\mathbf{J}} = \varepsilon\bar{\mathbf{g}}(\mathbf{J})$$

and for $\mathbf{P}$; the latter, in view of (9), takes the form

(6′) $$\dot{\mathbf{P}} = \varepsilon\bar{\mathbf{g}}(\mathbf{P}) + \mathbf{R}.$$

Since the difference between the right sides is of order $\lesssim \varepsilon^2$ (cf. (7)), for time $t \lesssim 1/\varepsilon$ the difference $|\mathbf{P} - \mathbf{J}|$ between the solutions is of order ε (Figure 226). On the other hand, $|\mathbf{I} - \mathbf{P}| = \varepsilon|\mathbf{k}| \lesssim \varepsilon$. Thus, for $t \lesssim 1/\varepsilon$, the difference $|\mathbf{I} - \mathbf{J}|$ is of order $\lesssim\varepsilon$, as was to be proved. □

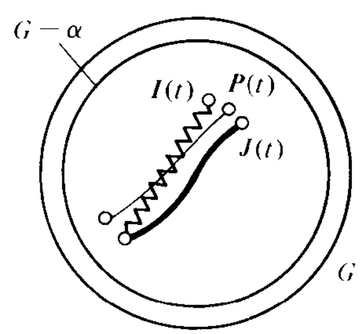

Figure 226 Proof of the theorem on averaging

To find an accurate estimate, we introduce the quantity

(10) $$\mathbf{z}(t) = \mathbf{P}(t) - \mathbf{J}(t).$$

Then (6′) and (9) imply

$$\dot{\mathbf{z}} = \varepsilon(\bar{\mathbf{g}}(\mathbf{P}) - \bar{\mathbf{g}}(\mathbf{J})) + \mathbf{R} = \varepsilon\frac{\partial\bar{\mathbf{g}}}{\partial\mathbf{P}}\mathbf{z} + \mathbf{R}',$$

where $|\mathbf{R}'| < c_2\varepsilon^2 + c_5\varepsilon|\mathbf{z}|$ if the segment $(\mathbf{P}, \mathbf{J})$ lies in $G - \alpha$. Under this assumption we find

(11) $$|\dot{\mathbf{z}}| < c_6\varepsilon|\mathbf{z}| + c_2\varepsilon^2 \quad (\text{where } c_6 = c_5 + c_1)$$
$$|\mathbf{z}(0)| < c_3\varepsilon.$$

Lemma. *If* $|\dot{\mathbf{z}}| \le a|\mathbf{z}| + b$ *and* $|\mathbf{z}(0)| < d$ *for* $a, b, d, t > 0$, *then* $|\mathbf{z}(t)| \le (d + bt)e^{at}$.

PROOF. $|\mathbf{z}(t)|$ is no greater than the solution $y(t)$ of the equation $\dot{y} = ay + b$, $y(0) = d$. Solving this equation, we find $y = Ce^{at}$, $\dot{C}e^{at} = b$, $\dot{C} = e^{-at}b$, $C(0) = d$, $C \le d + bt$. □

Now from (11) and the assumption that the segment $(\mathbf{P}, \mathbf{J})$ lies in $G - \alpha$ (Figure 226), we have

$$|\mathbf{z}(t)| < (c_3\varepsilon + c_2\varepsilon^2 t)e^{c_6\varepsilon t}.$$

From this it follows that, for $0 \le t \le 1/\varepsilon$,

$$|\mathbf{z}(t)| < c_7\varepsilon \qquad c_7 = (c_3 + c_2)e^{c_6}.$$

We see that, if $\alpha = d/3$ and ε is small enough, the entire segment $(\mathbf{P}(t), \mathbf{J}(t))(t \le 1/\varepsilon)$ lies inside $G - \alpha$ and, therefore,

$$|\mathbf{P}(t) - \mathbf{J}(t)| < c_8\varepsilon \quad \text{for all } 0 \le t \le \frac{1}{\varepsilon}.$$

On the other hand, $|\mathbf{P}(t) - \mathbf{I}(t)| < |\varepsilon\mathbf{k}| < c_3\varepsilon$. Thus, for all t with $0 \le t \le 1/\varepsilon$,

$$|\mathbf{I}(t) - \mathbf{J}(t)| < c_9\varepsilon \qquad c_9 = c_8 + c_3 > 0$$

and the theorem is proved. □

E *Adiabatic invariants*

Consider a hamiltonian system with one degree of freedom, with hamiltonian function $H(p, q; \lambda)$ depending on a parameter λ. As an example, we can take a pendulum:

$$H = \frac{p^2}{2l^2} + lg\frac{q^2}{2};$$

as the parameter λ we can take the length l or the acceleration of gravity g. Suppose that the parameter changes slowly with time. It turns out that in the limit as the rate of change of the parameter approaches 0, there is a remarkable asymptotic phenomenon: two quantities, generally independent, become functions of one another.

Assume, for example, that the length of the pendulum changes slowly (in comparison with its characteristic oscillations). Then the amplitude of its oscillation becomes a function of the length of the pendulum. If we very slowly increase by a factor of two the length of the pendulum and then very slowly decrease it to the original value, then at the end of this process the amplitude of the oscillation will be the same as it was at the start.

Furthermore, it turns out that the ratio of the energy H of the pendulum to the frequency ω changes very little under a slow change of the parameter, although the energy and frequency themselves may change a lot. Quantities such as this ratio, which change little under slow changes of parameter, are called by physicists *adiabatic invariants.*

It is easy to see that the adiabatic invariance of the ratio of the energy of a pendulum to its frequency is an assertion of a physical character, i.e., it is untrue without further assumptions. In fact, if we vary the length of a pendulum arbitrarily slowly, but chose the phase of oscillation under which

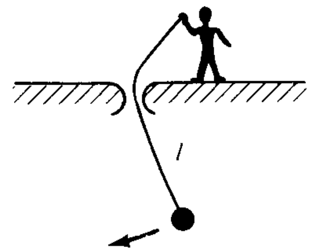

Figure 227 Adiabatic change in the length of a pendulum

the length increases and decreases, we can set the pendulum swinging (parametric resonance). In view of this, physicists have suggested formulating the definition of adiabatic invariance as follows: the person changing the parameters of the system must not see what state the system is in (Figure 227). Giving this definition a rigorous mathematical meaning is a very delicate and as yet unsolved problem. Fortunately, we can get along with a surrogate. The assumption of ignorance of the internal state of the system on the part of the person controlling the parameter may be replaced by the requirement that the change of parameter must be smooth, i.e., twice continuously differentiable.

More precisely, let $H(p, q; \lambda)$ be a fixed, twice continuously differentiable function of λ. Set $\lambda = \varepsilon t$ and consider the resulting system with slowly varying parameter $\lambda = \varepsilon t$:

$$(*) \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p}, \qquad H = H(p, q; \varepsilon t).$$

Definition. The quantity $I(p, q; \lambda)$ is an *adiabatic invariant* of the system (*) if for every $\kappa > 0$ there is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ and $0 < t < 1/\varepsilon$, then

$$|I(p(t), q(t); \varepsilon t) - I(p(0), q(0); 0)| < \kappa.$$

Clearly, every first integral is also an adiabatic invariant. It turns out that every one-dimensional system (*) has an adiabatic invariant. Namely, the adiabatic invariant is the action variable in the corresponding problem with constant coefficients.

Assume that the phase trajectories of the system with hamiltonian $H(p, q; \lambda)$ are closed. We define a function $I(p, q; \lambda)$ in the following way. For fixed λ there is a phase portrait corresponding to the hamiltonian function $H(p, q; \lambda)$ (Figure 228). Consider the closed phase trajectory passing through a point (p, q). It bounds some region in the phase plane. We denote the area of this region by $2\pi I(p, q; \lambda)$. $I = \text{const}$ on every phase trajectory (for given λ). Clearly, I is nothing but the action variable (cf. Section 50).

Theorem. *If the frequency $\omega(I, \lambda)$ of the system (*) is nowhere zero, then $I(p, q; \lambda)$ is an adiabatic invariant.*

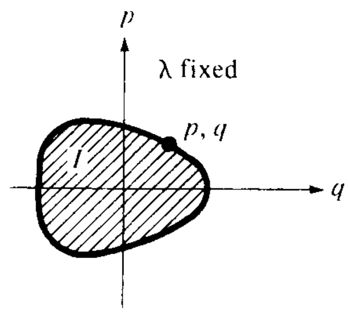

Figure 228 Adiabatic invariant of a one-dimensional system

F *Proof of the adiabatic invariance of action*

For fixed λ we can introduce action-angle variables I, φ into the system (*) by a canonical transformation depending on λ: $p, q \to I, \varphi$; $\dot{\varphi} = \omega(I, \lambda)$, $\dot{I} = 0$; $\omega(I, \lambda) = \partial H_0/\partial I$, $H_0 = H_0(I, \lambda)$.

We denote by $S(I, q; \lambda)$ the (multiple-valued) generating function of this transformation:

$$p = \frac{\partial S}{\partial q} \qquad \varphi = \frac{\partial S}{\partial I}.$$

Now let $\lambda = \varepsilon t$. Since the change from variables p, q to variables I, φ is now performed by a *time dependent* canonical transformation, the equations of motion in the new variables I, φ have the hamiltonian form, but with hamiltonian function (cf. Section 45A)

$$K = H_0 + \frac{\partial S}{\partial t} = H_0 + \varepsilon \frac{\partial S}{\partial \lambda}.$$

PROBLEM. Show that $\partial S(I, q; \lambda)/\partial \lambda$ is a single-valued function on the phase plane.

Hint. S is determined up to the addition of multiples of $2\pi I$.

In this way we obtain the equations of motion in the form

$$\dot{\varphi} = \omega(I, \lambda) + \varepsilon f(I, \varphi; \lambda) \qquad f = \frac{\partial^2 S}{\partial I\, \partial \lambda},$$

$$\dot{I} = \varepsilon g(I, \varphi; \lambda) \qquad g = -\frac{\partial^2 S}{\partial \varphi\, \partial \lambda},$$

$$\dot{\lambda} = \varepsilon$$

Since $\omega \neq 0$, the averaging theorem (Section 52C) is applicable. The averaged system has the form

$$\dot{J} = \varepsilon \bar{g} \qquad \dot{\Lambda} = \varepsilon.$$

But $g = (\partial/\partial\varphi)(\partial S/\partial\lambda)$, and $\partial S/\partial\lambda$ is a single-valued function on the circle $I = \text{const}$. Therefore, $\bar{g} = (2\pi)^{-1} \int g\, d\varphi = 0$, and in the averaged system J does not change at all: $J(t) = J(0)$.

By the averaging theorem, $|I(t) - I(0)| < c\varepsilon$ for all t with $0 \le t \le 1/\varepsilon$, as was to be proved. □

EXAMPLE. For a harmonic oscillator (cf. Figure 217),

$$H = \frac{a^2}{2} p^2 + \frac{b^2}{2} q^2 \qquad I = \frac{1}{2\pi} \pi \frac{\sqrt{2h}}{a} \frac{\sqrt{2h}}{b} = \frac{h}{\omega}, \qquad \omega = ab,$$

i.e., the ratio of energy to frequency is an adiabatic invariant.

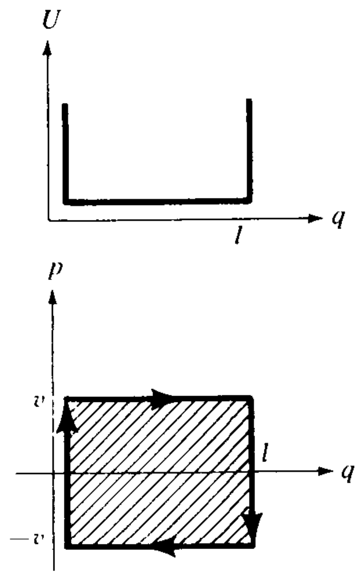

Figure 229 Adiabatic invariant of an absolutely elastic ball between slowly changing walls

PROBLEM. The length of a pendulum is slowly doubled ($l = l_0(1 + \varepsilon t)$, $0 \le t \le 1/\varepsilon$). How does the amplitude q_{max} of the oscillations vary?

Solution. $I = \frac{1}{2} l^{3/2} g^{1/2} q_{max}^2$; therefore,

$$q_{max}(t) = q_{max}(0) \left(\frac{l(0)}{l(t)} \right)^{3/4}.$$

As a second example, consider the motion of a perfectly elastic rigid ball of mass 1 between perfectly elastic walls whose separation l slowly varies (Figure 229). We may consider that a point is moving in an "infinitely deep rectangular potential well," and that the phase trajectories are rectangles of area $2vl$, where v is the velocity of the ball. In this case the product vl of the velocity of the ball and the distance between the walls turns out to be an adiabatic invariant.[95] Thus if we make the walls twice as close together, the velocity of the ball doubles, and if we separate the walls, the velocity decreases.

[95] This does not formally follow from the theorem, since the theorem concerns smooth systems without shocks. The proof of the adiabatic invariance of vl in this system is an instructive elementary problem.