NEW INSIGHTS ON THE STABILITY PROBLEM FROM RECENT RESULTS IN CLASSICAL PERTURBATION THEORY

by

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1. Overview

According to Poincaré (1892), the general problem of Dynamics is the study of a canonical system with Hamiltonian

$$H(p, q, \varepsilon) = H_0(p) + \varepsilon H_1(p, q), \tag{1.1}$$

where $p \equiv (p_1, \ldots, p_n) \in \mathcal{G} \subset \mathbf{R}^n$, with $\mathcal{G}$ an open set, are the action variables, $q \equiv (q_1, \ldots, q_n) \in \mathbf{T}^n$ are the angle variables, and $\varepsilon \in \mathbf{R}$ is a (small) parameter. The Hamiltonian is assumed to be an analytic function of p, q and ε.

The aim of the present lectures is to discuss some general results concerning the dynamics of such a kind of systems. Particular attention will be paid to a recent result of perturbation theory, namely the Nekhoroshev's theorem on stability over an exponentially large time scale.

The starting point of these lectures concerns the general dynamical behaviour of an integrable Hamiltonian system. Here, one should first agree on what is meant by “integrable”. The classical approach, for example, consists in looking for *integrability by quadratures*, namely in searching for a solution of a system of differential equations which involves only a finite number of algebraic operations and computation of integrals of known functions. More recently, one tends to consider an Hamiltonian system to be integrable if its phase space is foliated into invariant tori carrying either periodic or quasi periodic motions. Thus, the typical integrable system is represented by the Hamiltonian (1.1) for $\varepsilon = 0$, namely an Hamiltonian $H_0(p)$ depending only on the actions. The behaviour of such a system is illustrated in sects. 2.1 to 2.3.

A classical theorem concerning integrable systems is due to Liouville (Liouville, 1855; see also Whittaker, 1970, §148). The statement is that if a Hamiltonian system with n degrees of freedom admits n prime integrals which are independent and in involution (namely, the Poisson bracket between any two of them vanishes), then the system can be integrated by quadratures. A more recent version of this theorem, due to Arnold (1963), states that in such a case one can find action–angle variables such that the Hamiltonian turns out to depend on the actions only, and thus the motion is either periodic or quasiperiodic on a torus. The geometric ideas underlying this theorem will be the subject of sect. 2.4.

The naive approach to perturbation theory consists in trying to prove that the behaviour of the system (1.1) for $\varepsilon \neq 0$ is not too different from that of the unperturbed one, described by $H_0(p)$. More precisely, one tries to show that the phase space still admits a continuous foliation into invariant tori, which are close to the unperturbed ones, and that the motion is still either periodic or quasiperiodic on such tori. Essentially two methods are available to this end. The first one is the normal form theory: one looks for a canonical transformation $(p, q) = \mathcal{C}_\varepsilon(p', q')$, periodic in the angles, such that the transformed Hamiltonian $H'(p', q') = H(\mathcal{C}_\varepsilon(p', q'))$ is independent of the angles q', so that the system is directly seen to be integrable. The second method, which is more direct, consists in searching for prime integrals which are perturbations of the integrals of the unperturbed Hamiltonian $H_0(p)$. The theorem of Liouville and Arnold makes these two approaches equivalent, at least in principle. In fact, the latter method, although being more direct, presents some problems of formal consistency which are quite delicate, and have been solved only in particular cases; these difficulties are instead overcome by the normal form method, which is more general. Nevertheless, I will avoid here the technical apparatus of the canonical transformations, since the rigorous treatment of the method of the direct construction of prime integrals is definitely simpler, and so more convenient from a didactical viewpoint.

Luckily, the naive approach to perturbation theory fails: the theorem of Poincaré (1892) on the nonexistence of analytic prime integrals shows that, generically, the system (1.1) is not integrable (in the above sense). Such a result is quite delicate. That it should be at most a generic result is evident, since it is not difficult to construct examples of perturbed systems which are still integrable. On the other hand one could imagine, at first sight, to escape from the conclusions of Poincaré by considering a suitably restricted, but still relevant, class of Hamiltonians; such an attempt might even appear, from a purely formal viewpoint, to be successful, but a rigorous analysis shows that, generically, one just gets in such a way a nonconvergent series. The theorem of Poincaré will be the subject of sects. 3.1 to 3.3.

The next very important result is the theorem of Kolmogorov, Arnold and Moser (Kolmogorov, 1954; Moser, 1962; Arnold,1963), which originated the so called KAM theory. Let me briefly illustrate that theorem in order to make a connection between the classical work of Poincaré and the recent results of Nekhoroshev. Although a deep understanding of the KAM theory cannot be achieved without mastering the technical elements entering the proof, it is nevertheless possible to give at least a rough idea of it. The main point is that one renounces to have information about the flow on the whole phase space. Having realized that the existence of resonances among the frequencies is the key of the negative result of Poincaré, one considers those invariant tori of the unperturbed system $H_0(p)$ which are characterized by strongly nonresonant frequencies, in a sense that will be made more precise in sect. 2.3. The powerful result of the KAM theorem is that these tori do persist under small perturbations, being simply deformed. Thus, there exist initial data leading to quasiperiodic motions for the perturbed system. The natural question is then "how many" initial data produce such a kind of motion. From the viewpoint of measure theory, the answer is that the majority of the initial data lie on invariant tori, at least for small perturbation; the topological aspect however is not so plain: the invariant tori form a nowhere dense set in the phase space, and their complement on a surface of constant energy is connected if $n > 2$ (see sect. 2.3). The resulting picture of the motion is the following. For the majority of the initial data the motion is quasi periodic, and the orbit lies on a torus which is close to the unperturbed one. For systems with $n = 2$ the orbits starting in the gap between two invariant tori is confined inside that gap (taking into account the conservation of total energy), so that all the orbits stay forever close to an unperturbed torus. For systems with $n > 2$, instead, the orbits starting in the complement of the invariant tori can in principle go very far from the corresponding initial unperturbed torus. Such a phenomenon has been called "Arnold diffusion".

Whether or not Arnold diffusion is a real phenomenon is still an open question. An example, in a sense the simplest one, of a Hamiltonian system exhibiting such a diffusion has been produced by Arnold (1964) (Arnold, 1964; Arnold–Avez, 1967, §23). Such an example looks rather artificial; however, Arnold himself notes: "In contradistinction to stability, nonstability is itself stable. I believe that the mechanism of *transition chains* which guarantees that nonstability in our example is also applicable to the general case (for example, to the problem of three bodies)" (Arnold, 1964). The Arnold's example will be the subject of sect. 3.4.

The final topic, concerning the theorem of Nekhoroshev (1971, 1977, 1979), is in a sense complementary to KAM theory. Instead of restricting the set of initial data, one looks for results which hold for finite, but large, time intervals, and for all the orbits starting in an open set of initial data. The goal is to prove that the effect of the Arnold diffusion, if any, is so small that it can be distinguished from that of the deformation of the invariant tori only after an exceedingly large time; here, large time means that such a time grows exponentially with the inverse of the perturbation parameter ε.

This forces some change in our concept of stability. In the usual mathematical definition, stability is a property which involves *infinite* times. In this sense, for instance, the stability of the solar system can be proven only for the majority of initial data, according to KAM theory; analogously, the stability of the Lagrangian points L_4 and L_5 of the restricted circular problem of three bodies can be rigorously proven, in the framework of KAM theory, only in the planar case (with $n = 2$), while it holds only for the majority of the initial data in the spatial case. In Nekhoroshev's approach one asks instead for stability, namely the property usually required in the mathematical definition, only up to a finite time; the relevant condition is that such a time must be larger than any physically relevant time interval for the system considered. For example, if one considers the solar system, that time should be larger than the age of the solar system itself, or possibly the age of the universe. The exponential dependence of the estimated stability time on the inverse of the perturbation parameter, which is typical of the Nekhoroshev's like results ensures that such a time can actually be reached. This will be the subject of sect. 4.

2. Integrable systems

The present section is devoted to the study of integrable systems, namely a system of the type (1.1) with $\varepsilon = 0$. The periodic and the quasi periodic motions on a torus are illustrated in sect. 2.1, and sects. 2.2–2.3 describe the geometrical structure of the phase space in connection with the resonances. Sect. 2.4 contains the scheme of proof of the theorem of Liouville and Arnold on integrable systems. Finally, sect. 2.5 shows how this theorem can be applied in order to build up the action variables in the case of motion in a Keplerian potential.

2.1. Periodic and quasi periodic flow

Let us start our analysis by considering the system (1.1) with $\varepsilon = 0$, namely a Hamiltonian $H = H_0(p)$, independent of the angles $q_1, \ldots, q_n$. Such a system is trivially integrable. Indeed, denoting $\omega_l(p) = \frac{\partial H_0}{\partial p_l}(p)$, the Hamilton's equations are

$$\dot{q}_l = \omega_l(p) \ , \quad \dot{p}_l = 0 \ , \quad 1 \le l \le n \ , \tag{2.1}$$

and the solutions corresponding to the initial conditions $q_l(0) = q_{l,0}$ and $p_l(0) = p_{l,0}$ are

$$q_l(t) = \omega_l(p_{l,0})t + q_{l,0} \ , \quad p_l(t) = p_{l,0} \ . \tag{2.2}$$

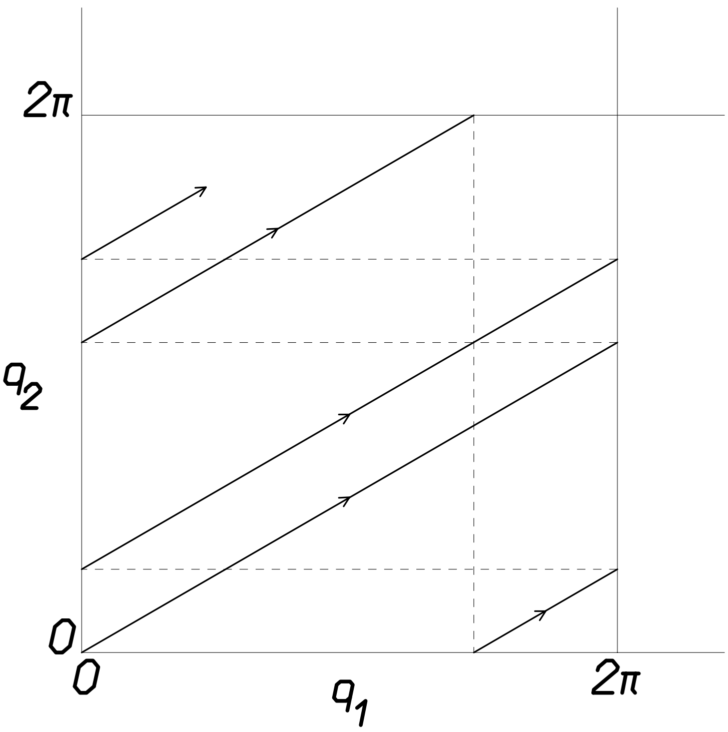

Fig. 1. *Plane representation of the quasi periodic flow on a two dimensional torus. The frequencies are* $\omega_1 = 1$, $\omega_2 = 1/\sqrt{3}$.

This completely solves the problem. However, it is interesting to discuss in more detail the geometrical aspects concerning the flow.

The equations (2.1) mean that the system admits n prime integrals, namely the actions $p_1, \ldots, p_n$. Thus, the phase space $\mathcal{G} \times \mathbf{T}^n$ admits a continuous foliation in invariant tori, which are parameterized by the action variables themselves, and the flow on each torus is characterized by the frequency vector $\omega_1(p), \ldots, \omega_n(p)$. So, one is led to investigate in more detail the flow on a torus. To do this, consider the case $n = 2$, and represent the torus $\mathbf{T}^2$, as usual, by the square of size 2π in the q_1, q_2 plane (see fig. 1). Without loss of generality, we can take $q_{1,0} = q_{2,0} = 0$, so that the solution (2.2) is represented on the plane by the straight line $\omega_2 q_1 - \omega_1 q_2 = 0$, and on the torus by reducing $(\bmod 2\pi)$ the coordinates of all points of that line. Consider now the successive intersections of the solution with the side $q_1 = 0$ of the torus: they are clearly represented by the sequence

$$0\,,\ \alpha \,(\bmod 2\pi)\,,\ 2\alpha \,(\bmod 2\pi)\,, \ldots,\ s\alpha \,(\bmod 2\pi)$$

with $\alpha = 2\pi\omega_2/\omega_1$. Thus, the qualitative aspects of the flow on the torus are well represented by a map of the circle into itself defined by a rotation of an angle α (see fig. 2). It is well known that the sequence $\{s\alpha \,(\bmod 2\pi)\}$ is periodic if and only if $\alpha/(2\pi)$ is a rational number, while for irrational $\alpha/(2\pi)$ such a sequence is dense on the circle (see for example Arnold–Avez, 1967, appendix 1). So, the orbit (2.2) on the torus is periodic for rational ω_2/ω_1 and dense on the torus otherwise.

In order to extend these results to the case $n > 2$ it is convenient to introduce the *resonance module* $\mathcal{M}_\omega \subset \mathbf{Z}^n$, defined as

$$\mathcal{M}_\omega = \{k \in \mathbf{Z}^n : k \cdot \omega = 0\} \tag{2.3}$$

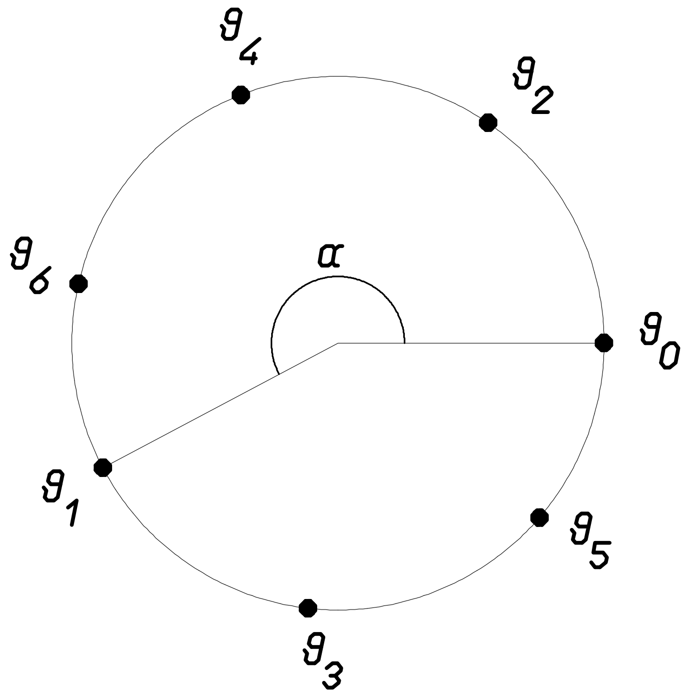

Fig. 2. *Map of the circle into itself defined by a rotation of an angle α. The sequence of points represented here corresponds to the intersection of the orbit of fig. 1 with the axis q_1.*

(here, the notation $k \cdot \omega = \sum_l k_l \omega_l$ has been used). The fact that $\mathcal{M}_\omega$ has the algebraic structure of a module directly follows from the definition. The dimension $\dim \mathcal{M}_\omega$ is sometimes called the *multiplicity of the resonance*, and is the number of independent resonance relations that exist among the frequencies $\omega_1, \ldots, \omega_n$. The extreme case are $\dim \mathcal{M}_\omega = 0$, which is usually called the *nonresonant* case, and $\dim \mathcal{M}_\omega = n - 1$, the completely resonant case. In the former case the orbit (2.2) is everywhere dense on the torus; in the latter one the torus is filled up by distinct strictly periodic orbits. In the intermediate case, any orbit is dense in a submanifold of the torus of dimension $n - \dim \mathcal{M}_\omega$.

As an example, consider the motion of a point mass in a central Keplerian field (in a bounded state). Since the problem is three dimensional, one has three frequencies, corresponding to the radial oscillation, the oscillation with respect to the equatorial plane of a fixed reference system, and the revolution around the center. All these frequencies actually coincide, and indeed this is the reason why all the orbits are strictly periodic, according to the first Kepler's law.

2.2. Isochronous and nonisochronous systems

Let us now come back to the consideration of the whole phase space. A natural question is whether one can find further prime integrals, involving the angles $q_1, \ldots, q_n$ (we ask, of course, for global prime integrals). To this end, one must distinguish between *isochronous* and *nonisochronous* systems (see for example Gallavotti, 1984).

The paradigm Hamiltonian of an isochronous system is

$$H(p) = \sum_l \omega_l p_l \ , \tag{2.4}$$

where $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R}^n$ are nonvanishing constants; this represents a system of harmonic oscillators. In such a case the geometric structure of the phase space is, in a sense, trivial, since all the invariant tori have the same frequencies. It is quite easy to get convinced that one can find $r = \dim \mathcal{M}_\omega$ global independent prime integrals which are also independent of the actions, since for example $\Phi_k = \sin(k \cdot q)$ with $k \in \mathcal{M}_\omega$ is such an integral; thus, the existence of resonances among the frequencies produces new prime integrals.

The nonisochronous case is well illustrated by the paradigm Hamiltonian

$$H(p) = \frac{1}{2} \sum_l p_l^2 \ . \tag{2.5}$$

In this case the frequencies $\omega_l = p_l$, $1 \le l \le n$, actually depend on the actions, so that an arbitrarily small change in the initial data can significantly change the topology of the orbits on the invariant torus. Thus, it is not so surprising that regular prime integrals involving the angles cannot be found. A formal proof goes as follows (see Poincaré, 1892, §82). Let $\Phi(p, q)$ be an analytic prime integral for $H(p)$; since it must be a periodic function of the angles $q_1, \ldots, q_n$, it can be expanded in a Fourier series as

$$\Phi(p, q) = \sum_{k \in \mathbf{Z}^n} \varphi_k(p) e^{ik \cdot q} \ ,$$

and since it is a prime integral the equation

$$\{H, \Phi\} = i \sum_k (k \cdot \omega(p)) \, \varphi_k(p) e^{ik \cdot q} = 0$$

must be satisfied. This implies that for any $k \in \mathbf{Z}^n$ either $\varphi_k(p) = 0$ or $k \cdot \omega(p) = 0$ must hold identically in p. If $\varphi_k(p) = 0$ for all $k \neq 0$ the integral depends only on the actions; otherwise there is at least one $k \neq 0$ with $k \cdot \omega(p) = 0$. Then, by differentiation with respect to p, one finds

$$\sum_{l=1}^n k_l \frac{\partial \omega_l}{\partial p_j} = 0 \ , \quad 1 \le j \le n \ ,$$

and this can hold only if the relation

$$\det \left(\frac{\partial \omega_l}{\partial p_j} \right) = 0$$

is satisfied. Thus, it is enough to assume the *nondegeneracy condition*

$$\det \left(\frac{\partial^2 H_0}{\partial p_l \partial p_j} \right) \neq 0$$

in order to ensure that no prime integrals involving the angles do exist. In particular, the nondegeneracy condition means that the frequencies $\omega(p)$ can be used, at least locally, as coordinates in the action space $\mathcal{G}$.

2.3. A result from diophantine theory

Let us now concentrate on nonisochronous systems. The fact that the frequencies $\omega_1(p), \ldots, \omega_n(p)$ can be used as coordinates allows one to introduce a kind of *geography of resonances* on the action domain $\mathcal{G}$. A resonance relation $k \cdot \omega(p) = 0$ with $0 \neq k \in \mathbf{Z}^n$ defines a resonant submanifold $\mathcal{G}$ of dimension $n-1$; more generally, a resonance module $\mathcal{M}$ generates a corresponding resonant manifold of dimension $n - \dim \mathcal{M}$. These resonant manifolds are clearly dense in $\mathcal{G}$. Such a fact is a key point in proving the theorem of Poincaré on the nonexistence of analytic prime integrals for the system (1.1).

From the viewpoint of modern perturbation theory it is also interesting to investigate the measure of the nonresonant points with respect to that of the resonant ones. The problem is stated more precisely as follows. Let us say that a frequency vector $\omega \in \mathbf{R}^n$ is strongly nonresonant in case one can find a positive function ψ such that one has

$$|k \cdot \omega| \geq \psi(|k|) \quad \text{for } 0 \neq k \in \mathbf{Z}^n \ ; \tag{2.6}$$

here the notation $|k| = \sum_l |k_l|$ has been used. Given an open bounded subset $\mathcal{D} \subset \mathbf{R}^n$, the question is whether one can determine ψ in such a way that the subset of the strongly nonresonant frequencies in $\mathcal{D}$, namely the set

$$\Omega = \{\omega \in \mathcal{D} : |k \cdot \omega| \geq \psi(|k|)\} \ ,$$

has large measure in $\mathcal{D}$.

A simple procedure to determine such a ψ is the following. Pick a nonzero $k \in \mathbf{Z}^n$, and consider

$$\tilde{\Omega}_k = \{\omega \in \mathcal{D} : |k \cdot \omega| < \psi(|k|)\} \ ,$$

namely the set of the ω's which are close to resonance with k. Such a set is contained in the set of points whose distance from the plane through the origin orthogonal to k is less than $\sqrt{n}\psi(|k|)/|k|$, intersected with $\mathcal{D}$ (the factor $\sqrt{n}$ is just due to the relation $|k| \leq \sqrt{n}\|k\|$, where $\|\cdot\|$ is the euclidean norm). Thus, its measure $\mu(\tilde{\Omega}_k)$ is bounded by

$$\mu(\tilde{\Omega}_k) \leq 2\sqrt{n}C\frac{\psi(|k|)}{|k|} \ ,$$

where C is a constant which depends only on the domain $\mathcal{D}$; an upper bound for C is $(\operatorname{diam} \mathcal{D})^{n-1}$. Then the measure of the complement of Ω in $\mathcal{D}$ cannot exceed

$$\mu\Bigl(\bigcup_{k\neq 0} \tilde{\Omega}_k\Bigr) \leq \sum_{k\neq 0} \mu(\tilde{\Omega}_k) \leq 2\sqrt{n}C \sum_{k\neq 0} \frac{\psi(|k|)}{|k|} \ .$$

Writing now

$$\sum_{k\neq 0} \frac{\psi(|k|)}{|k|} = \sum_{s>0} \sum_{|k|=s} \frac{\psi(s)}{s} \ ,$$

and using the fact that the number of vectors $k \in \mathbf{Z}^n$ satisfying $|k| = s$ does not exceed $2^n s^{n-1}$, one finally gets

$$\mu\Big(\bigcup_{k\neq 0} \tilde{\Omega}_k\Big) \le 2^{n+1}\sqrt{n}C \sum_{s>0} s^{n-2}\psi(s) \ .$$

Then it is enough to choose $\psi(s) = \gamma s^{-\tau}$ with suitable constants $\gamma > 0$ and $\tau > n-1$ in order to get that the complement of Ω in $\mathcal{D}$ has a measure which is small with γ. Such a result, although obtained with rough estimates, is optimal for what concerns the value of τ. Indeed, for $\tau < n-1$ the set Ω is empty, while for $\tau = n-1$ the set Ω is nonempty, but has zero measure (see for example Rüssmann, 1975). One can only improve the value of the constant γ, in particular the dependence of γ on n.

The result of the present section is at the very basis of the rigorous methods in perturbative theory. In particular, the KAM tori which persist under perturbation are precisely those with frequencies satisfying the condition (2.6).

2.4. The theorem of Liouville and Arnold

Let's now consider a generic canonical system with Hamiltonian $H(x, y)$; here, x and y are canonically conjugate variables, which are not restricted to be action–angle variables. The theorem of Liouville (1855) and Arnold (1963) is stated as follows.

Theorem 2.1: *Let $H(x, y)$ be the Hamiltonian of an autonomous canonical system with n degrees of freedom, and assume that the system admits n independent prime integrals $\Phi_1, \Phi_2, \ldots, \Phi_n$, with $\Phi_1 = H$, which are in involution; assume moreover the condition*

$$\det\left(\frac{\partial \Phi_i}{\partial y_k}\right) \neq 0 \ . \tag{2.7}$$

Consider the manifold M_α defined implicitly by $\Phi_1 = \alpha_1, \ldots, \Phi_n = \alpha_n$. Then:

i. M_α is invariant for each of the canonical flows generated by $\Phi_1, \ldots, \Phi_n$ respectively;

ii. if M_α is connected then it is diffeomorphic to the cartesian product $\mathbf{T}^r \times \mathbf{R}^{n-r}$ of r tori and $n-r$ lines; if M_α is also compact then it is diffeomorphic to a torus $\mathbf{T}^n$;

iii. if M_α is compact, then in a neighbourhood of M_α one can introduce action–angle variables such that the Hamiltonian is independent of the angles.

The statement i. is an easy consequence of the fact that $\Phi_1, \ldots, \Phi_n$ are in involution. Indeed, each manifold $\Phi_j = \alpha_j$, $1 \le j \le n$ is invariant for the flow generated by each function $\Phi_l, 1 \le l \le n$, considered as the Hamiltonian of a canonical system.

The statement ii. is more interesting. Let's first consider the local aspect. The n vector fields generated by $\Phi_1, \ldots, \Phi_n$ at any point $P \in M_\alpha$ are linearly independent, in particular non vanishing, and do commute (since $\Phi_1, \ldots, \Phi_n$ are independent and in involution). Denote now by $g_1^{t_1}, \ldots, g_n^{t_n}$ the flows generated by $\Phi_1, \ldots, \Phi_n$ respectively, and let $t \equiv (t_1, \ldots, t_n) \in V_0$, where V_0 is a neighbourhood of the origin of $\mathbf{R}^n$; then, for any $P_0 \in M_\alpha$ the flow $g^t(P_0) \equiv (g_1^{t_1} \cdot \ldots \cdot g_n^{t_n})(P_0)$ defines a local diffeomorphism between V_0 and a neighbourhood U of P_0 (the flows

$g_1^{t_1}, \ldots, g_n^{t_n}$ can be applied in any order, since they commute, so that to any $t \in V_0$ there corresponds a unique point $g^t(P_0) \in U$, and the local inversibility is guaranteed by the linear independence of the fields). Thus, the flows of $\Phi_1, \ldots, \Phi_n$ allow to build local coordinates in a neighbourhood of any point of M_α.

Let's now come to the global aspect. What has been said for a neighbourhood V_0 of the origin of $\mathbf{R}^n$ clearly holds for a neighbourhood V_s of any point $s \in \mathbf{R}^n$; thus, by composition, the point $g^t(P_0) \in M_\alpha$ is defined for any $t \in \mathbf{R}^n$; moreover, given any $P \in M_\alpha$, one can find a $t \in \mathbf{R}^n$ such that $g^t(P_0) = P$ (simply connect P to P_0 by a curve and compose the local flows of a finite family of neighbourhoods which covers the curve). This defines a global map $g^t : \mathbf{R}^n \to M_\alpha$. Let now G be the set of points $t \in \mathbf{R}^n$ such that $g^t(P_0) = P_0$. The set G clearly does not depend on P_0; moreover it is a group (more precisely G is a subgroup of $\mathbf{R}^n$), since $0 \in G$ (g^0 is the identity) and $t + s \in G$ for any $t, s \in G$, and contains only isolated points (due to the local inversibility of g^t). One then shows that there exist $r \le n$ linearly independent points $e_1, \ldots, e_r \in G$ which are a basis for G, in the sense that G coincides with the set $\{m_1 e_1 + \ldots + m_r e_r, (m_1, \ldots, m_r) \in \mathbf{Z}^r\}$. Consider now a basis $\{e_1, \ldots, e_r, \tau_1, \ldots, \tau_{n-r}\}$ of $\mathbf{R}^n$, where $\{e_1, \ldots, e_r\}$ is a basis of G; this allows in fact to define a coordinate system on M_α, which makes it diffeomorphic to the cartesian product $\mathbf{T}^r \times \mathbf{R}^n$. The case $r = 0$ corresponds to M_α diffeomorphic to $\mathbf{R}^n$; if instead M_α is compact, then one has $r = n$, so that M_α is a n–dimensional torus. In the latter case one has implicitly defined angular coordinates on the torus.

Let us finally come to the statement iii.; first one shows that the considerations above can be extended to a neighbourhood of the torus M_α. More precisely, letting $\alpha = (\alpha_1, \ldots, \alpha_n)$ to vary in an open domain $G \subset \mathbf{R}^n$, one proves that there exists a n–dimensional family of invariant tori; the Φ's itself and the angular coordinates defined above then make the neighbourhood of the torus diffeomorphic to $G \times \mathbf{T}^n$. These coordinates are not canonical; however one can express the original coordinates (x, y) as $x = f(\Phi, \psi), y = g(\Phi, \psi)$, where $\psi = (\psi_1, \ldots, \psi_n) \in \mathbf{T}^n$ are the angular coordinates describing a torus.

The construction of the action–angle variables is based on the consideration of the differential form

$$\mathrm{d}S = \sum_i y_i \mathrm{d}x_i \ , \tag{2.8}$$

where $y_1, \ldots, y_n$ are defined as functions of $\Phi_1, \ldots, \Phi_n, x_1, \ldots, x_n$ by inverting the relations $\Phi_1(x, y) = \alpha_1, \ldots, \Phi_n(x, y) = \alpha_n$. The involution condition on the Φ_j's implies that the differential form above is locally integrable. The action variables $p_1, \ldots, p_n$ are then defined as

$$p_j = \frac{1}{2\pi} \oint_{\gamma_j} \sum_l y_l \mathrm{d}x_l \ , \tag{2.9}$$

where $\gamma_1, \ldots, \gamma_n$ are n independent cycles on the torus M_α, namely γ_j is the cycle obtained by letting the angular coordinate ψ_j defined above to vary from 0 to 2π, while keeping ψ_k fixed for $k \neq j$; the fact that (2.8) is an exact differential makes the actual value of ψ_k irrelevant, so that $p_1, \ldots, p_n$ turn out to depend only on the torus, namely on $\Phi_1, \ldots, \Phi_n$. Thus we have n new prime integrals, $p_1, \ldots, p_n$

for the Hamiltonian H; moreover, assuming that the relation $p_j = p_j(\Phi_1, \ldots, \Phi_n)$, $(1 \le j \le n)$ can be inverted to give $\Phi_l = \Phi_l(p_1, \ldots, p_n)$, one has that the Hamiltonian H can be expressed as a function of $p_1, \ldots, p_n$ only. The angle variables $q_1, \ldots, q_n$, conjugated to $p_1, \ldots, p_n$, are finally introduced via the canonical transformation defined by the generating function

$$S(p, x) = \int_{P_0}^{P} \sum_j y_j \mathrm{d}x_j \ ,$$

where $y_1, \ldots, y_n$ are expressed as functions of (p, x).

2.5. An example: the Keplerian motion

As we have seen, the action–angle variables for an integrable system can be built up provided one is able to find the independent cycles on the torus which enter the definition (2.9) of the action variables. This is indeed the real difficulty in making explicit the action variables. However this is quite easy if the system is also separable, as is shown, for instance, by the case of the Keplerian motion.

The Hamiltonian of a mass point moving in a spherically symmetric field generated by a fixed center can be written

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin\vartheta^2}\right) + V(r) \ ; \tag{2.10}$$

here m is the mass of the point, r, ϑ, φ are spherical coordinates and $p_r, p_\vartheta, p_\varphi$ the corresponding momenta. In the Keplerian case one has also

$$V(r) = -\frac{k}{r} \ , \tag{2.11}$$

k being a constant; however, most of the following discussions does not depend on the form of $V(r)$, but only on the fact that it is independent of ϑ and φ.

The Hamiltonian (2.10) admits the well known prime integrals

$$\begin{aligned} \Gamma^2 &= p_\vartheta^2 + \frac{p_\varphi^2}{\sin\vartheta^2} \\ J &= p_\varphi \ , \end{aligned} \tag{2.12}$$

which, together with the Hamiltonian itself, are three independent prime integrals in involution.

Following the scheme of the theorem of Liouville and Arnold, let's consider the flow generated by the prime integrals. J is a trivially integrable Hamiltonian, and the variable φ conjugated to p_φ is already an angle; this gives the first cycle, γ_φ say (see fig. 3). Write now Γ^2 as

$$\Gamma^2 = p_\vartheta^2 + \frac{J^2}{\sin\vartheta^2} \ ;$$

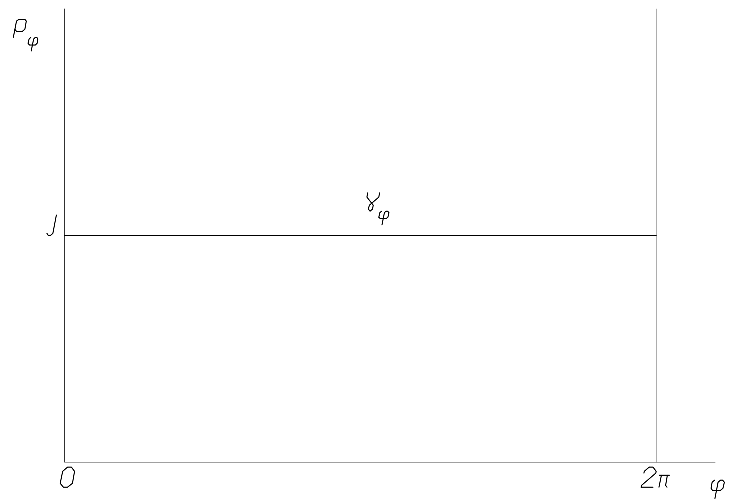

Fig. 3. *The cycle γ_φ for the problem of motion of a mass point in a central, spherically symmetric field.*

this is the Hamiltonian of a one–dimensional system, namely a point moving under the potential

$$V(\vartheta) = \frac{J^2}{\sin\vartheta^2} \ .$$

For any given value of $\Gamma^2 > \Gamma^2_{\min} = J^2$ the orbit in the phase plane ϑ, p_ϑ is a closed curve (see fig. 4), and this gives the second cycle, γ_ϑ say. Finally, write H as

$$H = \frac{1}{2m}\left(p_r^2 + \frac{\Gamma^2}{r^2}\right) + V(r) \ ,$$

so that it is in fact the Hamiltonian of a one–dimensional system with potential

$$V^*(r) = V(r) + \frac{\Gamma^2}{2mr^2} \ .$$

If $V^*(r)$ has a minimum, then there exists a set of values of H for which the orbit on the phase plane r, p_r is a closed curve, and this is the third cycle, γ_r say. For example, the case of the Keplerian potential (2.11) is represented in fig. 5. The cartesian product of the cycles $\gamma_r, \gamma_\vartheta, \gamma_\varphi$ is the three dimensional invariant torus.

Let's now come to the action variables. By inverting (2.12) and (2.10) with respect to the momenta, we get

$$\begin{aligned} p_r &= \left[2m(H - V(r)) - \frac{\Gamma^2}{r^2}\right]^{\frac{1}{2}} \\ p_\vartheta &= \left(\Gamma^2 - \frac{J^2}{\sin\vartheta^2}\right)^{\frac{1}{2}} \\ p_\varphi &= J \ , \end{aligned}$$

and by using the definition (2.9) of the action variables, namely by integrating the

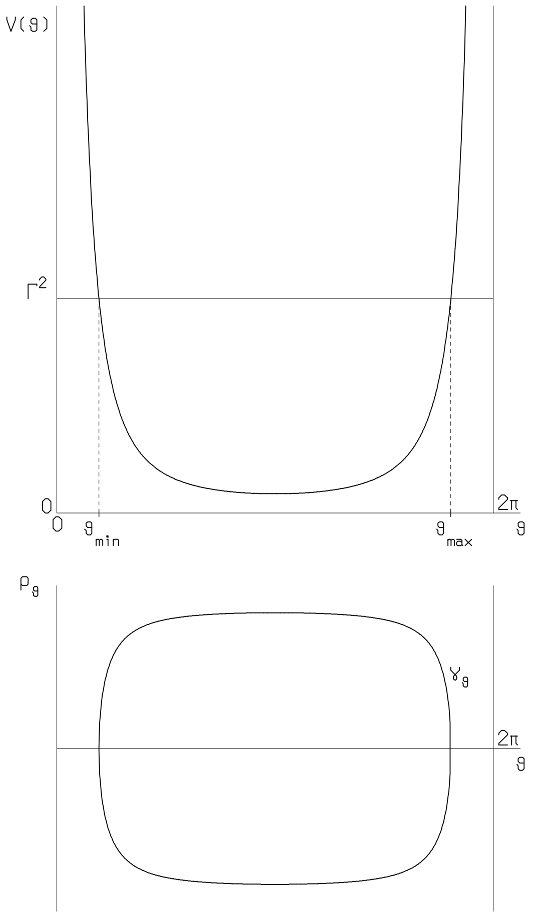

Fig. 4. *Construction of the cycle* γ_ϑ *for the problem of motion of a mass point in a central, spherically symmetric field.*

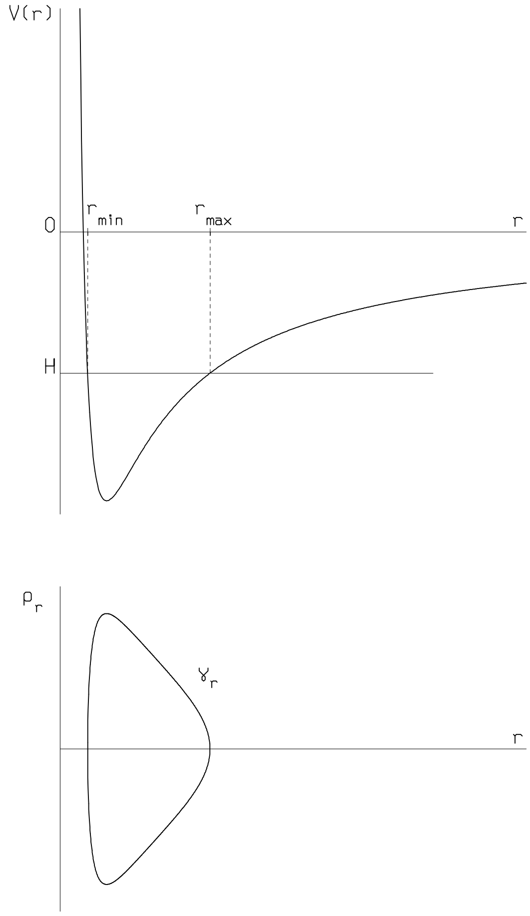

Fig. 5. *Construction of the cycle* γ_r *for the problem of motion of a mass point in a Keplerian field.*

differential form $p_r \mathrm{d}r + p_\vartheta \mathrm{d}\vartheta + p_\varphi \mathrm{d}\varphi$ over the cycles $\gamma_r, \gamma_\vartheta, \gamma_\varphi$, we compute

$$
\begin{aligned}
I_\varphi &= \frac{1}{2\pi}\int_0^{2\pi} J\mathrm{d}\varphi = J \\
I_\vartheta &= \frac{1}{\pi}\int_{\vartheta_{\min}}^{\vartheta_{\max}} \left(\Gamma^2 - \frac{J^2}{\sin\vartheta^2}\right)^{\frac{1}{2}} \mathrm{d}\vartheta = \Gamma - |J| \qquad (2.13)\\
I_r &= \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} \left[2m\left(H + \frac{k}{r}\right) - \frac{\Gamma^2}{r^2}\right]^{\frac{1}{2}} = -\Gamma + k\sqrt{-\frac{m}{2H}}\ .
\end{aligned}
$$

Here the first two integrals do not depend on the choice of the potential $V(r)$, while in the third one the explicit form of the Keplerian potential has been used. The values $\vartheta_{\min}, \vartheta_{\max}$ and $r_{\min}, r_{\max}$ are computed as the roots of the corresponding integrands in (2.13).

The Hamiltonian as a function of the actions is easily found to be

$$
H = -\frac{mk^2}{2(I_r + I_\vartheta + |I_\varphi|)^2}\ ,
$$

and looks similar to the Hamiltonian in Delaunay variables. This is due to the fact that the action–angle variables are not uniquely defined, since they depend on the choice of the independent cycles on the torus. The usual form is recovered by introducing the canonical momenta $L = I_r + I_\vartheta + |I_\varphi|$ and $G = I_\vartheta + |I_\varphi|$, and keeping I_φ (which is usually denoted by H when dealing with Delaunay's variables).

3. The theorem of Poincaré and the Arnold diffusion

The present section is devoted to some non–integrable aspects of the dynamics of a perturbed system, namely the system (1.1) for $\varepsilon \neq 0$. The theorem of Poincaré on the non–existence of analytic prime integrals is illustrated in sect. 3.1; in sect. 3.2 the genericity of the result of Poincaré is discussed, and sect. 3.3 shows how the problems of formal consistency leading to the negative results of Poincaré can be overcome in a particular but interesting case. This case will be treated by rigorous methods in sect. 4. Sect. 3.4 illustrates the example of diffusion due to Arnold.

3.1. The theorem of Poincaré

Consider now the complete system (1.1). With the aim of applying the results of sect. 2, we look for a power expansion in ε of a prime integral, namely

$$\Phi(p, q, \varepsilon) = \Phi_0(p, q) + \varepsilon\Phi_1(p, q) + \dots \ . \tag{3.1}$$

To this end we try to solve the equation

$$\{H, \Phi\} = 0$$

as follows. By substituting the expansions (1.1) and (3.1) for H and Φ, and equating term of the same order in ε we get the recursive system

$$\begin{aligned} \{H_0, \Phi_0\} &= 0 \\ \{H_0, \Phi_1\} &= -\{H_1, \Phi_0\} \\ &\cdots\cdots \\ \{H_0, \Phi_s\} &= -\{H_1, \Phi_{s-1}\} \ , \end{aligned} \tag{3.2}$$

that we can try to solve in order to get $\Phi_0, \Phi_1, \ldots$.

The first equation simply says that Φ_0 must be a prime integral for H_0; this is true, for example, if $\Phi_0 = \Phi_0(p_1, \ldots, p_n)$, namely does not depend on the angles. So, let's consider the generic form of the remaining equations. Since all the functions must be periodic in the angles, the r.h.s. of the generic equation for Φ_s can be expanded in Fourier series as

$$\sum_{k \in \mathbf{Z}^n} c_k(p) e^{ik\cdot q}$$

with known coefficients $c_k(p)$; assume now the same form for Φ_s, namely

$$\Phi_s(p, q) = \sum_{k \in \mathbf{Z}^n} \varphi_k(p) e^{ik\cdot q}$$

with unknown coefficients $\varphi_k(p)$. Then the equation (3.2) is transformed into

$$i\,(k\cdot\omega(p))\,\varphi_k(p) = c_k(p)\ ; \tag{3.3}$$

the formal solution of such an equation is simply

$$\varphi_k(p) = -i\frac{c_k(p)}{k\cdot\omega(p)}\ ,$$

but it is only valid if the denominator $k\cdot\omega(p)$ does not vanish. So one has to deal with the following problems:

i. *Consistency*: the average over the angles of the r.h.s. of eq. (3.2), namely the coefficient $c_0(p)$, must vanish. This is trivially true for $\{H_1,\Phi_0\}$, namely the r.h.s. of the equation for Φ_1, provided Φ_0 is a function of the actions only, but there is no apparent reason why this should be true a priori for all the equations;

ii. *Small denominators*: for $k\neq 0$, the expression $k\cdot\omega(p)$ vanishes in case of resonance; thus, unless the known coefficient $c_k(p)$ vanishes on the resonant manifold $k\cdot\omega(p)=0$, the system (3.2) can be solved only in a subset of the action domain $\mathcal{G}$ which excludes that manifold. Moreover, the same expression, even when non–vanishing, can assume arbitrarily small values, then raising doubts on the convergence of such a series.

The first problem can in fact be overcome: I will come back to this point later. The second one is instead at the very basis of the theorem of Poincaré.

Theorem 3.1: *Let $H(p,q,\varepsilon)=H_0(p)+\varepsilon H_1(p,q)$ with $p\in\mathcal{G}\subset\mathbf{R}^n$ and $q\in\mathbf{T}^n$ as in (1.1), and assume the following hypotheses:*

a. non degeneracy condition

$$\det\left(\frac{\partial^2 H_0}{\partial p_j\partial p_k}\right)\neq 0$$

b. genericity condition: no coefficient $h_k(p)$ in the Fourier expansion

$$H_1(p,q)=\sum_{k\in\mathbf{Z}^n} h_k(p)e^{ik\cdot q}$$

does identically vanish in $\mathcal{G}$.

Then there is no analytic prime integral $\Phi(p,q,\varepsilon)$ independent of the Hamiltonian.

The proof of this theorem requires three steps

i. Φ_0 must depend only on the action variables $p_1,\ldots,p_n$;

ii. The condition that Φ be independent of H is equivalent to the condition that Φ_0 be independent of H_0;

iii. Φ_0 cannot be independent of H_0.

The first statement directly follows from the non degeneracy condition as discussed in sect. 2.2. The proof of ii. proceeds as follows. Assume that there exists a prime integral Φ of the above form, with $\Phi_0=\Phi_0(H_0)$; then Φ_0 must be an analytic function of H_0. Consider now $\Psi=\Phi-\Phi_0(H)$, where $\Phi_0(H)$ is obtained by replacing H for H_0 in the explicit expression of Φ_0 as a function of H_0; then Ψ is clearly an analytic prime integral of H, which can be expanded as $\Psi(p,q) = \varepsilon\Psi_1(p,q) + \varepsilon^2\Psi_2(p,q) + \ldots$. Rename now Ψ_1 as Φ_0, Ψ_2 as Φ_1 and so on, and divide by ε; then one has a prime integral of the required form, so that the previous considerations can be applied again. Thus either we find, at some step, Φ_0 independent of H_0, or we simply expand the original prime integral Φ as a function of H. This proves ii. The proof of iii. makes explicit use of the small denominators. By using the Fourier expansion of H_1, the equation (3.3) for Φ_1 takes the more explicit form

$$(k \cdot \omega(p))\, \varphi_k(p) = \left(k \cdot \frac{\partial \Phi_0}{\partial p}(p)\right) h_k(p) \ .$$

In order to avoid zero divisors, the expression $k \cdot \frac{\partial \Phi_0}{\partial p}(p)$ must vanish at any point $p \in G$ where $k \cdot \omega(p)$ vanishes. Making reference to the resonance module $\mathcal{M}_\omega$ introduced in sect. 2.1, this is equivalent to saying that at any point $p \in G$ the gradients $\frac{\partial \Phi_0}{\partial p}(p)$ and $\omega(p) = \frac{\partial H_0}{\partial p}$ must be orthogonal to the same resonance module $M_{\omega(p)}$. In particular, if p is such that $\dim \mathcal{M}_{\omega(p)} = n-1$, then $\frac{\partial \Phi_0}{\partial p}(p)$ and $\omega(p) = \frac{\partial H_0}{\partial p}$ must be parallel, so that the rank of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial H_0}{\partial p_1}(p) & \ldots & \frac{\partial H_0}{\partial p_n}(p) \\ \frac{\partial \Phi_0}{\partial p_1}(p) & \ldots & \frac{\partial \Phi_0}{\partial p_n}(p) \end{pmatrix}$$

cannot be 2. Since the resonant points are dense in $\mathcal{G}$, this is true for any $p \in \mathcal{G}$ (by analyticity), and so Φ_0 cannot be independent of H_0. This concludes the proof.

3.2. Some remarks on the theorem of Poincaré

My aim is now to discuss in some detail the hypotheses of the theorem of Poincaré. Let me first discuss the genericity condition ii., which looks very strong. The following example shows that it can in fact be substantially weakened.

Consider the Hamiltonian

$$H(p,q,\varepsilon) = \frac{1}{2}(p_1^2 + p_2^2) + \varepsilon \left[\cos q_1 + \cos\,(q_1 - q_2) + \cos\,(q_1 + q_2) + \cos q_2\right]$$

and try to build a prime integral of the form (3.1) starting from $\Phi_0 = p_1$. Using the exponential form for the trigonometric functions, one computes

$$-\{H_1, p_1\} = -\frac{i}{2}\left[\left(e^{iq_1} + e^{i(q_1-q_2)} + e^{i(q_1+q_2)}\right) + \text{c.c.}\right]$$

(c.c. stays for complex conjugate of the previous parenthesis); so one easily gets

$$\Phi_1 = -\frac{1}{2}\left[\left(\frac{e^{iq_1}}{p_1} + \frac{e^{i(q_1-q_2)}}{p_1 - p_2} + \frac{e^{i(q_1+q_2)}}{p_1 + p_2}\right) + \text{c.c.}\right]$$

(one could add here an arbitrary function of p_1, p_2 that we have chosen equal to zero). Thus, it seems that we were able to build Φ_1, despite the theorem of Poincaré, at least in a domain which excludes the resonant manifolds $p_1 = 0$, $p_1 - p_2 = 0$ and $p_1 + p_2 = 0$. This is due to the fact that only a finite number of nonvanishing

Fourier components appear in H_1. If however one computes the r.h.s. of the equation for Φ_2 one sees* that the Poisson bracket $\{H_1, \Phi_1\}$ introduces, among others, the Fourier components $2q_1 - q_2$, $2q_1 + q_2$, $2q_1 - 2q_2$, and $2q_1 + 2q_2$, so that the domain of definition of Φ_2 must exclude, besides the previous ones, the further resonant manifolds $2p_1 - p_2 = 0$ and $2p_1 + p_2 = 0$. Proceeding in the same way, it is clear that the r.h.s. of the equation for Φ_s will contain practically all the Fourier components $k_1 q_1 + k_2 q_2$ with $0 < |k_1| + |k_2| \leq 2s$; thus, the perturbative procedure itself produces all the coefficients which are not present in H_1, and this shows that it is in fact impossible to build the whole expansion of the prime integral Φ.

The example above can be easily generalized. Suppose indeed that in the Hamiltonian (1.1) one has

$$H_1(p, q) = \sum_{|k| \leq N} h_k(p) e^{ik \cdot q}$$

(where $|k| = |k_1| + \ldots + |k_n|$) for some positive integer N, while $H_0(p)$ satisfies the non–degeneracy condition i.; then it is evident that the equation for Φ_s contains in general (almost) all the Fourier components with $|k| \leq sN$.

A trivial exception is the following one. Consider a module $\mathcal{M} \subset \mathbf{Z}^n$ with $\dim \mathcal{M} > 0$, and assume that H_1 contains no Fourier component $k \in \mathcal{M}$, i.e. take H_1 of the form

$$H_1(p, q) = \sum_{k \in \mathbf{Z}^n \setminus \mathcal{M}} h_k(p) e^{ik \cdot q} \ .$$

In this case it is easily checked that any function $\Phi = \sum_j \alpha_j p_j$, with $\alpha \in \mathbf{R}^n$ and $\alpha \perp M$ is a prime integral, and so one has $n - \dim M$ independent prime integrals. This is not enough to ensure integrability (unless $\dim M = n - 1$), but simply allows to reduce the Hamiltonian system to a lower dimensional one.

What has been said shows how far the genericity condition ii. of the theorem of Poincaré can be weakened without affecting the final result. Thus, the theorem appears to be applicable to a generic Hamiltonian satisfying the non–degeneracy condition. Let me stress here that the result of Poincaré is obtained on a purely formal basis: no convergence problem is raised, since there is no series whose convergence can be investigated.

The situation is well different if the non–degeneracy condition i. is removed. Such a case, which was considered by Whittaker (1916), Cherry (1924), Birkhoff (1927) and Contopoulos (1960), is discussed in the next section.

3.3. A formally integrable case

Consider a canonical system with Hamiltonian

$$H(x, y) = H_0(x, y) + H_1(x, y) + H_2(x, y) + \ldots \tag{3.4}$$

* I'm ignoring here the consistency problem quoted in sect. 3.1, namely the fact that the r.h.s. of eq. (3.2) could contain terms independent of q_1, q_2. One could prove that this does not happen.

where $(x, y) \in \mathbf{R}^{2n}$ are canonically conjugate variables,

$$H_0(x, y) = \frac{1}{2} \sum_{l=1}^{n} \omega_l (x_l^2 + y_l^2) \tag{3.5}$$

is the Hamiltonian of a system of harmonic oscillators with non–vanishing angular frequencies* $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R}^n$, and $H_s(x, y)$, for $s \geq 1$, is a homogeneous polynomial of degree $s + 2$ in the canonical variables.

The canonical transformation to the action–angle variables

$$x_l = \sqrt{2p_l} \cos q_l \ , \quad y_l = \sqrt{2p_l} \sin q_l, \quad 1 \leq l \leq n$$

gives H_0 the form (2.2) of an isochronous system; however, this introduces an unwanted singularity at the origin, due to the square root; so let's keep the x, y variables. The role of the perturbative parameter is played here by the size of the neighbourhood of the origin where the Hamiltonian is considered; thus, the perturbation order is determined by the degree of a homogeneous polynomial.

The translation of the scheme of sect. 3.1 is an easy matter. One looks for a prime integral

$$\Phi^{(l)}(x, y) = \Phi_0^{(l)}(x, y) + \Phi_1^{(l)}(x, y) + \ldots \tag{3.6}$$

where $\Phi_0^{(l)}(x, y) = p_l = \frac{1}{2}(x_l^2 + y_l^2)$ is the action of the l–th oscillator, and $\Phi_s(x, y)$ is a homogeneous polynomial of degree $s + 2$. Thus one finds the system of equations

$$\begin{aligned} \{H_0, \Phi_1^{(l)}\} &= -\{H_1, \Phi_0^{(l)}\} \\ &\cdots\cdots \\ \{H_0, \Phi_s^{(l)}\} &= -\{H_1, \Phi_{s-1}^{(l)}\} - \ldots - \{H_s, \Phi_0^{(l)}\} \ . \end{aligned} \tag{3.7}$$

The algebraic aspect of the above equations is quite simple. The unperturbed Hamiltonian H_0 acts as a linear operator $L_{H_0} = \{H_0, .\}$ from the linear space, $\mathcal{P}^{(r)}$ say, of homogeneous polynomials of a fixed degree r into itself. Moreover, using the complex canonical coordinates $(\xi, \eta) \in \mathbf{C}^{2n}$ defined by

$$x_l = \frac{1}{\sqrt{2}}(\xi_l + i\eta_l) \ , \quad y_l = \frac{i}{\sqrt{2}}(\xi_l - i\eta_l) \ , \quad 1 \leq l \leq n \tag{3.8}$$

one finds

$$H_0 = i \sum_l \omega_l \xi_l \eta_l \ , \tag{3.9}$$

so that the operator above takes a diagonal form; indeed, by applying it to a monomial $\xi^j \eta^k \equiv \xi_1^{j_1} \ldots \xi_n^{j_n} \eta_1^{k_1} \ldots \eta_n^{k_n}$ one gets

$$L_{H_0} \xi^j \eta^k = i((k - j) \cdot \omega) \xi^j \eta^k \ .$$

* Note that it is *not* requested that the frequencies be positive. Such a fact is relevant in discussing the stability of the origin.

Defining now, as usual, $\mathcal{R}$ as the image of $\mathcal{P}^{(r)}$ by L_{H_0}, one sees that the eq. (3.7) can be solved if the r.h.s. belongs to $\mathcal{R}$; on the other hand, by defining the null space $\mathcal{N}$ as the inverse image of the null element by L_{H_0}, one gets that both $\mathcal{N}$ and $\mathcal{R}$ are linear subspaces of the same space $\mathcal{P}^{(r)}$, which are disjoint, namely satisfy $\mathcal{N} \cap \mathcal{R} = \{\emptyset\}$, and generate $\mathcal{P}^{(r)}$ by direct sum, namely satisfy $\mathcal{N} \oplus \mathcal{R} = \mathcal{P}^{(r)}$. Thus the system (3.7) can be solved provided the r.h.s. has no component in $\mathcal{N}$: this is nothing but the consistency condition referred to in sect. 3.1.

The formal existence of the prime integral is stated by the following proposition (Diana et al., 1975; Giorgilli, 1989).

Proposition 3.2: *Let $H(x, y)$ be as in (3.5), and assume:*

i. non resonance: for $k \in \mathbf{Z}^n$ one has $k \cdot \omega = 0$ if and only if $k = 0$;

ii. reversibility: The Hamiltonian is an even function of the momenta, namely satisfies $H(x, -y) = H(x, y)$.

Then there exist n independent formal integrals $\Phi^{(1)}, \ldots, \Phi^{(n)}$ of the form (3.6) which are in involution.

The proof is based on two simple remarks. First, the nonresonance condition implies that any function $f \in \mathcal{N}$ must be even in the momenta, since it can depend only on the action variables $p_1 = \xi_1\eta_1, \ldots, p_n = \xi_n\eta_n$; next, it is easily seen that the Poisson bracket between functions of the same parity is odd, while the Poisson bracket between functions of different parity is even. Using these remarks, and proceeding by induction one sees that if $\Phi_s^{(l)}$ has been determined for $0 \le s \le r$ as an even function of the momenta (which is true for $r = 0$), then the r.h.s. of the equation for $\Phi_{r+1}^{(l)}$ is an odd function, so that it has no component in $\mathcal{N}$, and so $\Phi_{r+1}^{(l)}$ can also be determined; such a solution is unique up to an arbitrary term $\tilde{\Phi}_{r+1}^{(l)} \in \mathcal{N}$, and turns out to be an even function. This concludes the proof.

The proposition above is, as stressed, a formal one, in the sense that all the construction is performed by simply using algebra, regardless of the convergence of the series so generated. In the same spirit, one could apply the method of Liouville and Arnold to build the action–angle variables (see, for example Whittaker, 1970, §199). However, one should discuss the convergence properties of the series so generated. Indeed the denominators $k \cdot \omega$, although non vanishing, are not bounded from below. On the other hand, there are known examples of series involving small denominators which are convergent (see for example the discussion in Whittaker, 1970, §198). An essentially negative answer to the problem of convergence was given by Siegel (1941).

3.4. The Arnold's example of diffusion

As was briefly illustrated in sect. 1, the powerful results coming from KAM theory, namely the existence of invariant tori for a perturbed system, do not exclude the possibility of the Arnold diffusion. Here, my aim is to illustrate an active mechanism which can possibly generate that diffusion (Arnold, 1964; see also Arnold–Avez, 1967).

The basic elements of the mechanism are:

i. the whiskered torus;

ii. the transition torus;

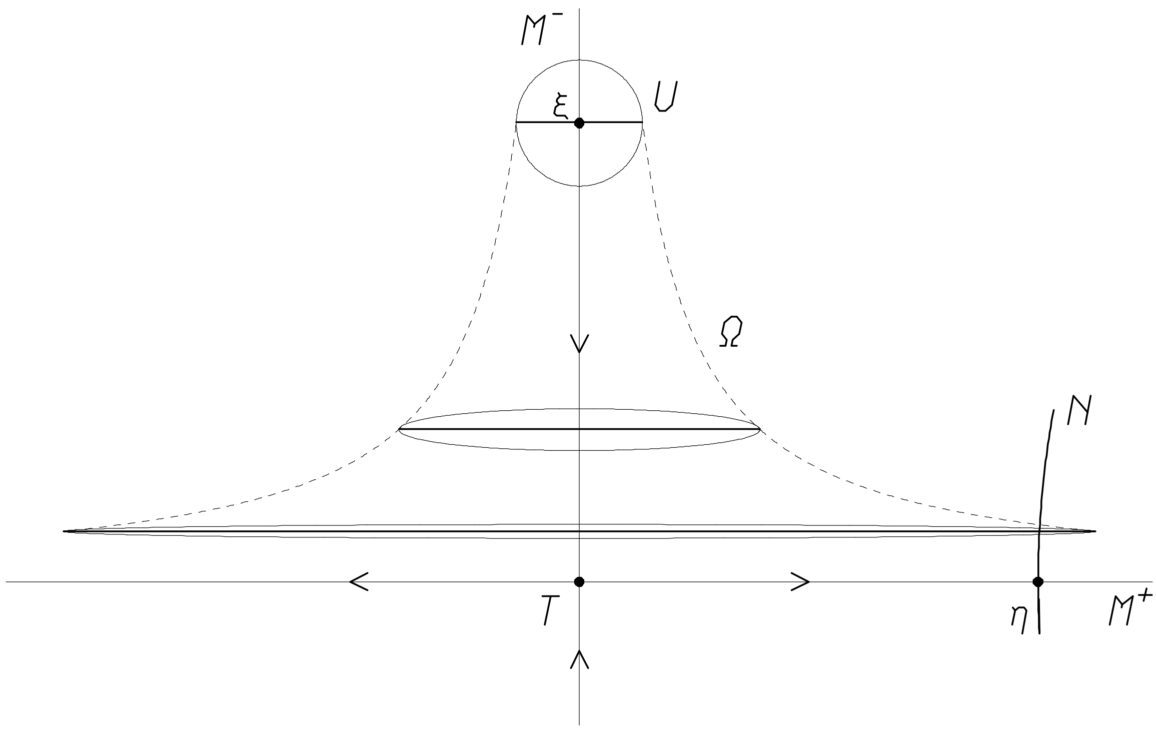

Fig. 6. *The transition torus: representation of the flow generated by the system (3.10) in the plane x, y. The torus T is the origin, and the axes represent the arriving whisker M^- and the departing whisker M^+. The set Ω of all the images of the domain U by the flow intersects any manifold N transversal to the departing whisker at an arbitrary point η.*

iii. the transition chain.

These elements are not intrinsically related to the property of the system of being Hamiltonian; the idea of Arnold was to build up an explicit example of Hamiltonian system which contains all these elements.

The whiskered torus is a generalization of a saddle equilibrium point for a system of differential equations in the plane. The simplest example is represented by the system of differential equations

$$\begin{aligned} \dot{x} &= \lambda x \\ \dot{y} &= -\mu y \\ \dot{z} &= 0 \\ \dot{\varphi} &= \omega \ , \end{aligned} \tag{3.10}$$

where $(x, y, z) \in \mathbf{R}^3$ and $\varphi \in \mathbf{T}^k$ are coordinates in the phase space, λ and μ are positive constants, and $\omega \in \mathbf{R}^k$ is the vector of frequencies, which are assumed to be nonresonant. The torus T_0 defined by $x = y = z = 0$ is clearly invariant, and the orbits of the system (3.10) are dense on it (actually, there is a one–dimensional family of such tori, parameterized by z). The manifolds M_0^- and M_0^+ defined by $x = z = 0$ and $y = z = 0$ respectively are also invariant, and their intersection is the torus T_0; moreover, an orbit starting from any point of M_0^- tends to T_0 for $t \to +\infty$, while an orbit starting from any point of M_0^+ tends to T_0 for $t \to -\infty$. Usually, M_0^- and M_0^+ are referred to as the *stable* and the *unstable* manifold respectively;

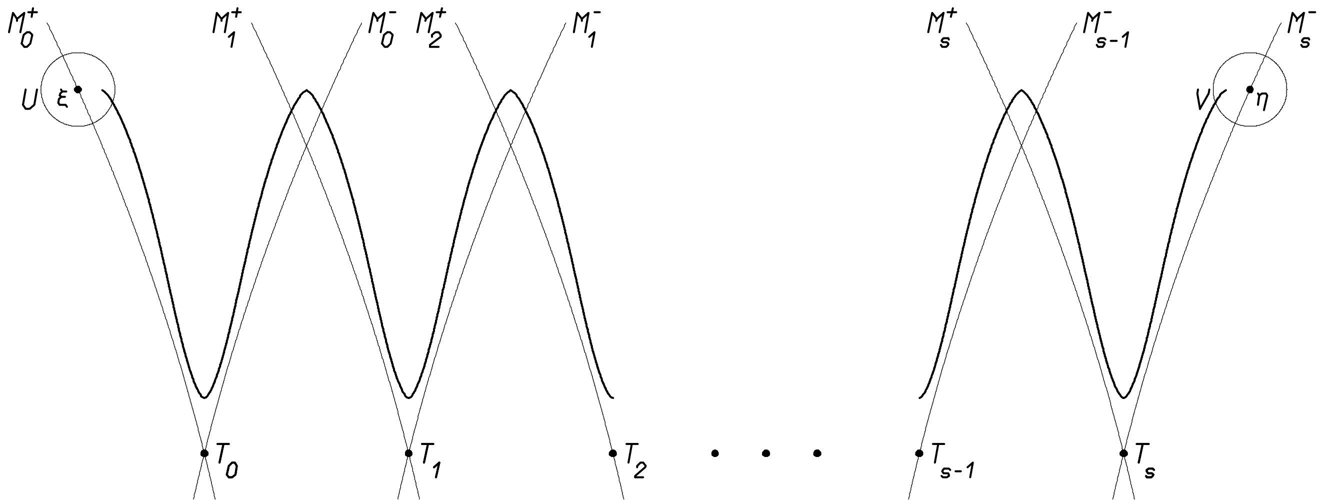

Fig. 7. *The transition chain: a chain of transition tori such that the departing whisker of a torus, T_j say, intersects transversally the arriving whisker of the next torus T_{j+1}. Then any neighbourhood U of an arbitrary point ξ of the arriving whisker of the first torus is connected by an orbit to any neighbourhood V of an arbitrary point η of the departing whisker of the last torus.*

however, let me follow Arnold, and call them the *arriving whisker* and the *departing whisker*. In general, a whiskered torus T is defined as the connected intersection of an arriving whisker with a departing whisker.

The same example (3.10) is useful to represent a transition torus. Let $\xi = (0, y_0, 0, \varphi) \in M_0^-$ and $\eta = (x_1, 0, 0, \varphi_1) \in M_0^+$ be arbitrary points of the arriving and the departing whisker respectively, and let U be an arbitrary neighbourhood of ξ. Denote by $\Omega = \bigcup_{t\geq 0} U(t)$ the set of all points of all the orbits starting in U. Then Ω intersects any manifold N which is transverse to M_0^+ at η. This is illustrated (unfortunately in a somehow misleading way) by fig. 6. The subset $V \subset U$ of the points with $y = y_0$ generates, due to the flow, a set of surfaces $V(t)$ which are parallel to M_0^+ and converge to M_0^+ for $t \to +\infty$; on the other hand, due to the fact that the frequencies ω are nonresonant, there exists a sequence $\{t_i\}_{i\geq 1}$ such that $\varphi_0 + \omega t_i \to \varphi_1$. The corresponding sequence $\{V(t_i)\}_{i\geq 1}$ of surfaces generated by V intersects N for i large enough. In general, a transition torus is a whiskered torus satisfying the property that the images of an arbitrary neighbourhood of an arbitrary point $\xi \in M^-$ of its arriving whisker intersect any manifold N which is transverse to the departing whisker M^+ at an arbitrary point $\eta \in M^+$.

A transition chain is a set $T_0, \ldots, T_s$ of transition tori with the further property that the departing whisker M_j^+ of the transition torus T_j intersects transversally the arriving whisker M_{j+1}^- of the next transition torus T_{j+1} (see fig. 7). One proves then that an arbitrary neighbourhood U of an arbitrary point $\xi \in M_0^-$ is connected by an orbit to an arbitrary neighbourhood of an arbitrary point $\eta \in M_s^+$, i.e. there exists an orbit $\zeta(t)$ such that $\zeta(0) \in U$ and $\zeta(t) \in V$ for a certain t. To see this, it is enough to iterate the mechanism of the transition torus: the set Ω of the images of U by the flow intersects M_1^-, since M_1^- is transverse to M_0^+ at a point $\eta_1 \in M_0^+ \cap M_1^-$; let now $\xi_1 \in \Omega \cap M_1^-$, then there is a neighbourhood U_1 of ξ_1 such that $U_1 \subset \Omega$; the set $\Omega_1 \subset \Omega$ of the images of U_1 by the flow then intersects M_2^-, and so on for s times.

The aim now is to build up an Hamiltonian system which has a chain of transition tori. The Arnold's suggestion is to consider the nonautonomous canonical system with Hamiltonian

$$H(p,q,t) = \frac{1}{2}(p_1^2 + p_2^2) + \varepsilon(\cos q_1 - 1)[1 + \mu(\sin q_2 + \cos t)] \ , \tag{3.11}$$

where $(p_1, p_2) \in \mathbf{R}^2$ and $(q_1, q_2, t) \in \mathbf{T}^3$ (i.e., the Hamiltonian is 2π–periodic in t), and $\varepsilon\,, \mu$ are real parameters. The corresponding equations are

$$\begin{aligned} \dot q_1 &= p_1 \\ \dot q_2 &= p_2 \\ \dot p_1 &= \varepsilon \sin q_1 [1 + \mu\,(\sin q_2 + \cos t)] \\ \dot p_2 &= \varepsilon\mu(1 - \cos q_1)\cos q_2 \ . \end{aligned} \tag{3.12}$$

Let's proceed step by step. For $\varepsilon = \mu = 0$ the system is clearly integrable and nonisochronous. For $\varepsilon > 0$ and $\mu = 0$ the system is still integrable, and has a one–parameter family of two dimensional invariant tori $p_1 = q_1 = 0$, parameterized by $\omega = p_2$ (the two dimensions are due to q_2, which evolves with frequency ω, and t, which has frequency 1). For irrational ω the torus T_ω is a whiskered torus, whose whiskers have equations

$$\begin{aligned} H^{(1)} &\equiv \frac{1}{2}p_1^2 + \varepsilon(\cos q_1 - 1) = 0 \\ H^{(2)} &\equiv \frac{1}{2}p_2^2 = \frac{1}{2}\omega^2 \ . \end{aligned}$$

Let now $\varepsilon > 0$ and $\mu > 0$, and write the equation for the whiskers of T_ω as

$$\begin{aligned} H^{(1)} &= \Delta_1^{\pm}(q_1, q_2, t, \omega) \\ H^{(2)} &= \frac{1}{2}\omega^2 + \Delta_2^{\pm}(q_1, q_2, t, \omega) \ . \end{aligned}$$

By considering two nearby whiskered tori T_ω and $T_{\omega'}$ with $\omega < \omega'$, we look for the intersection of the departing whisker M_ω^+ of T_ω with the arriving whisker $M_{\omega'}^-$ of $T_{\omega'}$. Setting $q_1 = \pi$, one has the equations

$$\begin{aligned} \Delta_1^+(\pi, q_2, t, \omega) &= \Delta_1^-(\pi, q_2, t, \omega') \\ \frac{1}{2}\omega^2 + \Delta_2^+(\pi, q_2, t, \omega) &= \frac{1}{2}\omega'^2 + \Delta_2^-(\pi, q_2, t, \omega') \ . \end{aligned} \tag{3.13}$$

The interesting fact now is that the functions $\Delta_{1,2}^{\pm}(\pi, q_2, t, \omega)$ and $\Delta_{1,2}^{\pm}(\pi, q_2, t, \omega')$ can be explicitly determined, up to a term $O(\mu^2)$. Indeed, one has

$$\Delta_1^+(\pi, q_2, t_0, \omega) = \mu \int_\infty^0 \{H, H^{(1)}\}\mathrm{d}(t - t_0) + O(\mu^2) \ ,$$

where the integral can be computed along the solution corresponding to the departing whisker of the unperturbed system, which is known. Analogous formulae hold for the other quantities. This allows one to prove that the equations (3.13) can be solved provided one has $|\omega - \omega'| \sim \mu \exp(-1/\sqrt{\varepsilon})$ and $\mu \sim \exp(-1/\sqrt{\varepsilon})$

Let finally $0 < \omega_0 < A < B$, and look for a chain $T_{\omega_0}, \ldots, T_{\omega_s}$ of transition tori with $\omega_s > B$; if $|\omega_j - \omega_{j-1}|$ is small enough such a chain is a transition chain, and so there is an orbit of the system (3.12) with $p_2(0) < A$ and $p_2(t) > B$ for a certain t. This proves the existence of diffusion for the Hamiltonian system (3.11)

A last consideration concerns the time. The fact that the transition tori must be very close in order to ensure that their separatrices intersect suggests that the time needed for the orbit to travel along the entire chain can be very large, for example of order $\exp(1/\sqrt{\varepsilon})$. The theorem of Nekhoroshev, which will be the subject of sect. 4, shows that the diffusion is really very slow.

4. A simple proof of the theorem of Nekhoroshev

The idea of looking for stability over large, although finite, times was already proposed by Moser (1955) and Littlewood (1959). In particular, the latter author applied such an idea to the Lagrangian points of the problem of three bodies. The general formulation is due to Nekhoroshev (1971, 1977, 1979). The original formulation of the theorem of Nekhoroshev holds for a generic Hamiltonian like (1.1), but involves a lot of technical elements, in particular a detailed analysis of the topology of resonances in the action space. The case I'm going to consider is the one discussed in sect. 3.3; although less general, it contains the essential ideas of Nekhoroshev's like theory.

4.1. Algebraic and analytic setting

Let us consider the canonical system with Hamiltonian

$$H(x,y) = H_0(x,y) + H_1(x,y) \ , \tag{4.1}$$

with

$$H_0(x,y) = \frac{1}{2}\sum_{l=1}^{n} \omega_l (x_l^2 + y_l^2) \ , \tag{4.2}$$

where $\omega_1, \ldots, \omega_n$ are constants and H_1 a homogeneous polynomial of degree 3. This is like (3.4); the fact that the perturbation is just a polynomial is not relevant: considering the full Hamiltonian (3.4) is just a technical fact (the general and detailed treatment can be found in Giorgilli, 1988). I will also assume that the Hamiltonian is an even function of the momenta and that there exists a nonincreasing sequence $\{\alpha_s\}_{s\geq 1}$ of positive real numbers such that the nonresonance condition

$$|k\cdot\omega| \geq \alpha_s \quad \text{for } k \in \mathbf{Z}^n \ , \quad 0 < |k| \leq s+2 \tag{4.3}$$

is satisfied. Thus, we can formally build the prime integrals $\Phi^{(l)}(x,y) = I_l + \Phi_1^{(l)} + \ldots$, $1 \leq l \leq n$, with $I_l = \frac{1}{2}(x_l^2 + y_l^2)$, as described in sect. 3.3. In particular, the solution will be made unique by the condition $\Phi_s^{(l)} \in \mathcal{R}$ for $s \geq 1$.

In order to make rigorous that scheme we must be able to evaluate the size of a function. To this end, let x, y be complex variables, and write a homogeneous polynomial, $f(x, y)$ say, of order s as

$$f(x,y) = \sum_{|j+k|=s} f_{j,k} x^j y^k \ ,$$

with coefficients $f_{j,k} \in \mathbf{C}$. Choose now a vector $R = (R_1, \ldots, R_n) \in \mathbf{R}^n_+$ of positive real numbers. The norm of f is then defined as

$$\|f\| = \sum_{j,k} |f_{j,k}| \, R^{j+k} \ . \tag{4.4}$$

Use will also be made of the quantity $\Lambda = (\min_l R_l)^{-1}$. Considering then a domain

$$\Delta_{\varrho R} = \bigl\{ (x,y) \in \mathbf{R}^{2n} \, : \ x_l^2 + y_l^2 \le \varrho^2 R_l^2 , \ 1 \le l \le n \bigr\} \ , \tag{4.5}$$

namely the cartesian product of n disks of radii ϱR_l in the coordinate plane x_l, y_l, the size of $f(x, y)$ in $\Delta_{\varrho R}$ is bounded by

$$|f(x,y)| \le \varrho^s \|f\|$$

(recall that f is a homogeneous polynomial of degree s).

4.2. Technical estimates

The equation (3.7) for the Hamiltonian (4.1) takes the simpler form

$$\{H_0, \Phi_s^{(l)}\} = \Psi_s^{(l)} \ , \tag{4.6}$$

with

$$\Psi_s^{(l)} = \begin{cases} -\{H_1, I_l\} & \text{for } s = 1 \\ -\{H_1, \Phi_{s-1}^{(l)}\} & \text{for } s > 1 \end{cases} \tag{4.7}$$

The aim now is to translate this recursive set of equations into a set of recursive estimates on the norms of the polynomials $\Phi_s^{(l)}$ and $\Psi_s^{(l)}$. This is given as follows. First transform the Hamiltonian to complex variables ξ, η, defined by the canonical transformation (3.8). Then build up the expansion in complex variables by recursively solving (4.6); this involves two basic operations: a Poisson bracket to determine $\Psi_s^{(l)}$, and the inversion of the linear operator $L_{H_0} = \{H_0, .\}$. Finally, transform back to real variables by the inverse of the transformation (3.8). We need to know how the norms are propagated through these operations. This is given by the following technical estimates.

i. The transformation (3.8) to complex variables changes the norm of a homogeneous polynomial of degree s at most by a factor $2^{s/2}$; the same holds for the inverse.

ii. The Poisson bracket between two homogeneous polynomials f and g of degree s and r respectively is estimated by

$$\|\{f,g\}\| \le sr\Lambda^2 \|f\| \|g\| \ ; \tag{4.8}$$

in particular one has also , for $1 \le l \le n$,

$$\|\{f, I_l\}\| \le s\|f\| \ . \tag{4.9}$$

iii. The equation (4.6) admits a solution $\Phi_s^{(l)}$ whose norm is bounded in complex variables by

$$\|\Phi_s^{(l)}\| \le \frac{1}{\alpha_s} \|\Psi_s^{(l)}\| \ , \tag{4.10}$$

with α_s satisfying (4.3).

The proof of i. is trivial, and (4.10) is just a consequence of the diagonal form of L_{H_0} in complex variables. To prove (4.8) write

$$\{f,g\} = \sum_{j,k,j',k'} f_{jk} g_{j'k'} x^{j+j'} y^{k+k'} \sum_{l=1}^{n} \frac{j_l k'_l - j'_l k_l}{x_l y_l} \ ,$$

and use the definition of the norm to compute

$$\begin{aligned} \|\{f,g\}\| &\le \sum_{j,k,j',k'} |f_{jk}|\, |g_{j'k'}|\, R^{j+j'+k+k'} \sum_{l=1}^{n} \frac{j_l k'_l + j'_l k_l}{R_l^2} \\ &\le sr\Lambda^2 \Bigl(\sum_{j,k} |f_{jk}|\, R^{j+k} \Bigr) \Bigl(\sum_{j',k'} |g_{j'k'}|\, R^{j'+k'} \Bigr) \\ &\le sr\Lambda^2 \|f\| \|g\| \ . \end{aligned}$$

The proof of (4.9) is essentially the same.

The estimates above allow us to get bounds on the prime integral. Indeed, starting from $\|H_3\| \le E$ in real variables, from (4.6) and (4.7) one finds the estimates

$$\left\| \Psi_s^{(l)} \right\| \le A_s \ , \quad \left\| \Phi_s^{(l)} \right\| \le \frac{A_s}{\alpha_s} \ , \quad s \ge 1 \ , \tag{4.11}$$

where $\{A_s\}_{s\ge 1}$ is a sequence of real numbers defined by

$$\begin{aligned} A_1 &= 24E \\ A_s &= 12E(12\Lambda^2 E)^{s-1} \frac{(s+1)!}{\prod_{l=1}^{s-1} \alpha_l} \ , \quad s \ge 1 \ . \end{aligned} \tag{4.12}$$

This is seen as follows. For $s = 1$ it is a trivial consequence of the technical estimates (4.9) and (4.10). For $s > 1$ we look for a recursive estimate: assuming that (4.11) holds up to $s-1$, one gets, by (4.7),

$$\|\Psi_s^{(l)}\| = \|\{H_1, \Phi_{s-1}^{(l)}\}\| \le 3(s+1)\Lambda^2 \mathcal{E} A_{s-1}/\alpha_{s-1} = A_s$$

with $\mathcal{E} = 2^{3/2}E$, and thus, by (4.10), $\|\Phi_s^{(l)}\| \le A_s/\alpha_s$. The numerical factors take into account the transformation to complex variables before starting the construction of the integrals, and the transformation back to real variables at the end.

4.3. Truncated integrals

The estimates (4.11) and (4.12) clearly do not allow to prove the convergence of the expansions of the prime integrals $\Phi^{(l)}$. This, of course, could be simply due to the fact that we are unable to find better estimates; recall however that the non–convergence, in general, of these prime integrals has been proven by Siegel. On the other hand, it is well known that the perturbative expansions are a commonly used method, particularly in Celestial Mechanics. So, let's truncate the procedure at some arbitrary order r, and consider the truncated prime integrals

$$\Phi^{(l,r)} = I_l + \Phi_1^{(l)} + \ldots + \Phi_r^{(l)} \tag{4.13}$$

whose time derivatives clearly are

$$\dot{\Phi}^{(l,r)} = -\{H_1, \Phi_r^{(l)}\} \ . \tag{4.14}$$

My aim is now to show that, even if we do not perform an explicit construction of the integrals $\Phi^{(l,r)}$, we can nevertheless obtain significant information from the technical estimates of the previous section. Indeed, suppose that, for a real system, we are only able to observe the values of the harmonic actions $I_1, \ldots, I_n$ as functions of time (this is, essentially, what we do when we compute the osculating elements of the orbit of a planet from the observed positions). The natural question here is how much these quantities, which are the approximate constants of our problem, can vary in time. Using the fact that the $\Phi^{(l,r)}$'s are, hopefully, better conserved than the I_l's, one finds the bound

$$|I_l(t) - I_l(0)| \le |I_l(t) - \Phi^{(l,r)}(t)| + |\Phi^{(l,r)}(t) - \Phi^{(l,r)}(0)| + |\Phi^{(l,r)}(0) - I_l(0)| \ . \tag{4.15}$$

In order to simplify the discussion, let me, just for a moment, set $r = 1$ and assume that the $\Phi^{(l,1)}$'s are exact integrals (this could, of course, be true for a particular Hamiltonian), so that the central term of (4.15) vanishes. The two remaining terms are estimated by noting that for any (x, y) in a domain $\Delta_{\varrho R}$ defined by (4.5) one has

$$\left| \left(\Phi^{(l,1)} - I_l \right) (x, y) \right| = \left| \Phi_1^{(l)}(x, y) \right| < \tilde{C} \varrho^3 \ ,$$

with a suitable constant $\tilde{C}$ (see below for a general explicit estimate). Thus, one has the bound

$$|I_l(t) - I_l(0)| < 2\tilde{C}\varrho^3 \tag{4.16}$$

for any t, provided one can guarantee that the orbit is confined for all times in $\Delta_{\varrho R}$. This, in turn, is true provided one has $I_l(t) < \frac{1}{2}\varrho^2 R_l^2$, namely provided the starting point (x_0, y_0) of the orbit at time $t = 0$ lies inside the domain $\Delta_{\varrho_0 R}$, with ϱ_0 satisfying

$$\frac{1}{2}\varrho_0^2 R_l^2 < \frac{1}{2}\varrho^2 R_l^2 - 2\tilde{C}\varrho^3 \ .$$

This requires that ϱ be small, precisely that $\varrho < \overline{\varrho} = \min_l [R_l^2/(4\tilde{C})]$. The variation in time of the harmonic actions $I_1, \ldots, I_n$ appears here to be clearly due to a *deformation* of the invariant surfaces, since one has $\Phi^{(l,1)}$ =const instead of I_l =const,

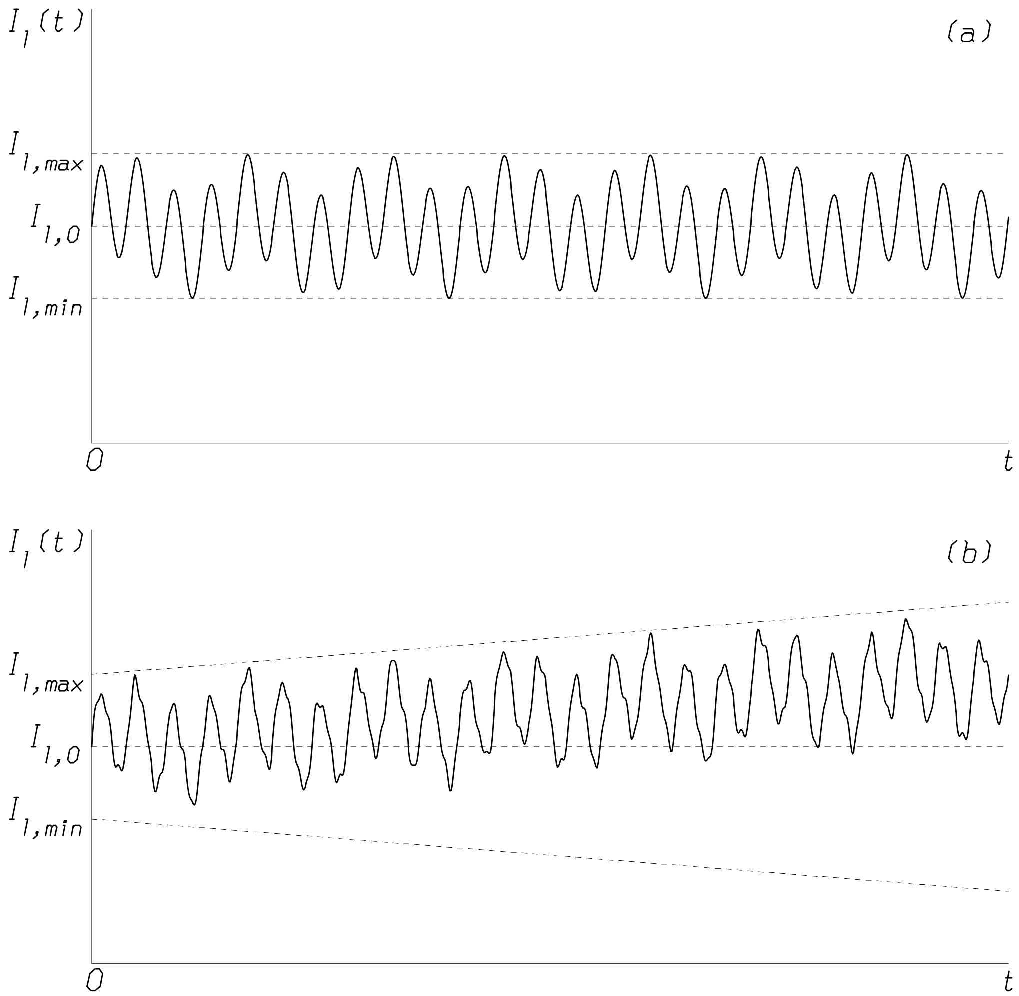

Fig. 8. *Representation of the time evolution of the harmonic action under the effect of the* <u>*deformation*</u> *and of the* <u>*noise*</u>*. (a): the deformation causes a quasi periodic, bounded oscillation of the action $I_l(t)$. (b): the noise adds further frequencies, and may cause a secular variation (drift), at most linear in time. The figure represents the worst case.*

as for the unperturbed case (i.e. with $H_1 = 0$). This is illustrated in fig. 8–(a): the action $I_l(t)$ exhibits a quasi periodic oscillation, but is confined in the strip $I_{l,\min} \le I_l(t) \le I_{l,\max}$, whose width is estimated by (4.16).

Let us now take into account the fact that the $\Phi^{(l,r)}$'s are not exact constants (but still let $r = 1$). Thus, by (4.14), the time derivative $\dot\Phi^{(l,1)}$ is a homogeneous polynomial of order 4, so that for $(x,y) \in \Delta_{\varrho R}$ one has

$$\left|\dot\Phi^{(l,1)}(x,y)\right| < C_1 \varrho^4 \ , \tag{4.17}$$

with a suitable constant C_1. Thus, superimposed to the deformation, there is a variation in time of $I_l(t)$ due to the dynamical evolution of $\Phi^{(l,1)}$; note however that such an evolution is quite slow with respect to the one due to the deformation, since $\dot I_l$ is of order of ϱ^3, i.e. much larger than (4.17). Following Nekhoroshev, let me refer to this effect as a *noise*. The situation is illustrated in fig. 8–(b): the disturbing term $-\{H_1, \Phi_1^{(l)}\}$ introduces new frequencies (recall the discussion in sect. 3.2), and moreover may cause a secular variation (*drift*) of $I_l(t)$. An *a priori* bound is given by the estimate (4.17), which guarantees that the width of the strip where $I_l(t)$ is confined increases at most linearly with t. The figure represents the worst case, in which the variation is actually linear. Taking into account such an effect, the bound (4.16) must be changed to

$$|I_l(t) - I_l(0)| < 2\tilde{C}\varrho^3 + C_1\varrho^4 t \ . \tag{4.18}$$

However, we still have to ensure that $I_l(t) < \frac{1}{2}\varrho^2 R_l^2$ (since all the estimates hold on $\Delta_{\varrho R}$). This, of course, cannot be true for all times, but we can use the fact that the noise has a small effect in order to ensure that it holds for a quite large time. To do this, it is enough, for example, to require $C_1\varrho^4 t \le 2\tilde{C}\varrho^3$ (i.e.,we do not allow $\left|\Phi^{(l,1)}(t) - \Phi^{(l,1)}(0)\right|$ to be larger than the deformation); this clearly holds provided

$$|t| \le T_1 = \frac{2\tilde{C}}{C_1} \cdot \frac{1}{\varrho} \ , \tag{4.19}$$

and provided the initial domain $\Delta_{\varrho_0 R}$ satisfies

$$\frac{1}{2}\varrho_0^2 R_l^2 < \frac{1}{2}\varrho^2 R_l^2 - 4\tilde{C}\varrho^3 \ . \tag{4.20}$$

Again, this requires the further condition

$$\varrho < \varrho_1 = \min_l \left(\frac{R_l^2}{4\tilde{C}}\right) \ ; \tag{4.21}$$

the quantity ϱ_1 here plays the role of a *threshold*, above which the perturbation theory is useless.

Extending the considerations above to a generic $r > 1$ is just a technical fact: it is enough to produce explicit bounds on the deformation $\left|\left(\Phi^{(l,r)} - I_l\right)(x,y)\right|$ and on the noise $|\dot{\Phi}^{(l,r)}(x,y)|$ for $(x,y) \in \Delta_{\varrho R}$; the first of these quantities can still be expected to be of order ϱ^3, while the latter one should admit a bound like $C_r\varrho^{r+3}$. The fact that the constant C_r may be explicitly estimated constitutes the main ingredient of the Nekhoroshev's like theory. The result is summarized by the following

Proposition 4.1: *Consider the Hamiltonian (4.1)–(4.2), and assume that the frequencies* $\omega_1, \ldots, \omega_n$ *satisfy the condition (4.3) for a suitable sequence* $\{\alpha_s\}_{s\ge1}$*; assume that for a given* $R \in \mathbf{R}^n_+$ *there exists* $E > 0$ *such that* $\|H_1\| \le E$ *with the norm (4.4). Then, for any integer* $r \ge 1$ *there exist* n *truncated prime integrals* $\Phi^{(l,r)} = I_l + \sum_{s=1}^{r} \Phi_s^{(l)}$*, with* $I_l = \frac{1}{2}(x_l^2 + y_l^2)$ *and* $\Phi_s^{(l)}$ *a homogeneous polynomial of degree* $s+2$*, such that one has that* $\dot{\Phi}^{(l,r)} = \{H, \Phi^{(l,r)}\}$ *is a homogeneous polynomial of degree* $r+3$*; moreover, for any positive* ϱ *and any* $(x,y) \in \Delta_{\varrho R}$ *with* $\Delta_{\varrho R}$ *defined by (4.5), one has the bounds*

$$\left|\left(\Phi^{(l,r)} - I_l\right)(x,y)\right| < \frac{24E}{\alpha_1}\varrho^3\left[1 - (\sigma_r\varrho)^r\right](1 - \sigma_r\varrho)^{-1} \tag{4.22}$$

$$\left|\dot{\Phi}^{(l,r)}(x,y)\right| < C_r \varrho^{r+3} \ , \tag{4.23}$$

where

$$\begin{aligned} \sigma_1 &= 1 \\ \sigma_r &= 12\Lambda^2 E \left[\frac{(r+1)!}{\prod_{l=2}^{r} \alpha_l}\right]^{\frac{1}{r-1}} \ , \quad r > 1 \\ C_r &= 3E\left(12\Lambda^2 E\right)^r \frac{(r+2)!}{\prod_{l=1}^{r} \alpha_l} \ , \quad r \ge 1 \ . \end{aligned} \tag{4.24}$$

I omit here the detailed proof, since it is a purely technical matter and gives no relevant informations for what follows. The interested reader can find all details in (Giorgilli, 1988).

4.4. Exponential estimates

The estimates of the proposition above exhibit a strong dependence on r, so that one expects, a priori, that the choice of r has a significant impact on the final result. Let me illustrate such a fact by making again reference to fig. 8. Since we are still considering a truncated integral, the qualitative picture does not significantly change with respect to the case $r = 1$; from a quantitative point of view instead the width of the interval $[I_{l,\min}, I_{l,\max}]$ and the slope of the dashed straight lines in fig. 8–(b) can considerably change with r.

Let me add a further remark. The truncation order r appears here as an extraneous element: it is clearly nonsense to introduce a "user defined parameter" in the statement of a stability result concerning a physical system. The actual choice of a quite low r is in fact dictated by the practical impossibility of performing an explicit expansion of a prime integral up to high orders. The interesting aspect of the theory developed up to now is instead that the constants which appear in the r.h.s. of the estimates (4.22) and (4.23) can be *explicitly* computed, since they depend only on the original Hamiltonian and on the choice of the domain. This suggests the possibility of sharpening the theory by looking for a choice of r which is, in some sense, the optimal one. This leads in a quite natural way to exponential estimates of Nekhoroshev's type.

Let's first consider the estimate (4.23). Recalling that $\{\alpha_s\}_{s\ge1}$ is a nonincreasing sequence, one realizes that the expression $C_r\varrho^r$, considered as a function of r for a fixed ϱ, has a minimum. Indeed, one clearly has

$$C_r\varrho^r = \frac{12\Lambda^2 E(r+2)}{\alpha_r}\varrho \cdot C_{r-1}\varrho^{r-1} \ ;$$

thus the estimate is improved by adding the order r if $12\Lambda^2 E(r+2)\varrho/\alpha_r < 1$.

A more explicit analytic estimate can be obtained if one knows more about the sequence $\{\alpha_s\}_{s\ge1}$. To this end, the natural choice is to make use of the result of sect. 2.3, namely to take

$$\alpha_s = \gamma(s+2)^{-\tau} \ , \quad s > 1$$

with suitable constants $\gamma > 0$ and $\tau \geq n-1$. Then the condition above becomes $12\Lambda^2 E(r+2)^{\tau+1}\varrho/\gamma < 1$, and one is naturally led to choose the optimal truncation order $r_{\rm opt} = r_{\rm opt}(\varrho)$ defined as

$$r_{\rm opt} = \left[\left(\frac{\varrho^*}{\varrho} \right)^{1/(\tau+1)} \right] - 2$$

(here, $[\cdot]$ denotes the integer part), with

$$\varrho^* = \frac{\gamma}{12\Lambda^2 E}\ . \tag{4.25}$$

The perturbation theory is then useful if

$$\varrho \leq 3^{-(\tau+1)} \varrho^*\ ,$$

since this ensures that one has $r_{\rm opt} \geq 1$. This allows one to remove r from the estimates of proposition 3.2, by substituting $r_{\rm opt}(\varrho)$ in place of r, so that the truncation order turns out to be determined by the size of the domain containing the initial data.

Let me now stress only the relevant steps, and skip the technical details. Write the estimate (4.23) as $(r!)^a (\varrho/\varrho^*)^r$, and minimize it by setting $r = (\varrho^*/\varrho)^{1/a}$. Then, using $r! \sim r^r e^{-r}$ (by Stirling's formula), one immediately gets

$$(r!)^a \left(\frac{\varrho}{\varrho^*} \right)^r \sim \exp\left[-a \left(\frac{\varrho^*}{\varrho} \right)^{1/a} \right]\ .$$

Thus, the noise becomes exponentially small with the inverse of the size of the domain. This is the heart of Nekhoroshev's like theory. The formal statement of the theorem is the following:

Theorem 4.2: *Consider the canonical system with Hamiltonian (4.1), and assume that the harmonic frequencies ω satisfy the nonresonance condition $|k\cdot\omega| > \gamma\, |k|^{-\tau}$ for $k \in \mathbf{Z}^n$, with real constants $\gamma > 0$ and $\tau \geq 0$; assume moreover that for a given $R \in \mathbf{R}^n_+$ there exists a real constant $E > 0$ such that $\|H_1\| \leq E$. Then for any $\varrho \leq 3^{-(\tau+1)}\varrho^*$, with ϱ^* given by (4.25), there exist n approximate integrals $\Phi^{(l)}(x,y)$ such that for $(x,y) \in \Delta_{\varrho R}$, defined by (4.5), one has*

$$\left| \left(\Phi^{(l)} - I_l \right)(x,y) \right| < \frac{8\cdot 3^\tau}{\Lambda^2 \varrho^*} \varrho^3$$

$$\left| \dot\Phi^{(l)}(x,y) \right| < 6 \left(\frac{e^2}{2} \right)^{\tau+1} E{\varrho^*}^3 \left(\frac{\varrho}{\varrho^*} \right)^{\frac{1}{2}} \exp\left[-(\tau+1) \left(\frac{\varrho^*}{\varrho} \right)^{1/(\tau+1)} \right]\ .$$

In view of this result, all the considerations made in the previous section on the truncated integrals may be repeated, with the additional information that the effect of the noise can be detected only after an exceedingly large time interval. Indeed, the estimate (4.13) of the stability time is changed to

$$|t| < T = T_* \exp\left[-(\tau+1) \left(\frac{\varrho^*}{\varrho} \right)^{1/(\tau+1)} \right]$$

with a suitable T_*. In the very words of Littlewood, "while not eternity, this is a considerable slice of it".

4.5. A note on the application to physical models

The theory developed up to now can be effectively used to study the dynamics of a system in the neighbourhood of an elliptic equilibrium. The classical example in Celestial Mechanics is the Lagrangian point L_4 (or L_5) of the problem of three bodies.

If one considers the Sun–Jupiter case, and assumes that Jupiter revolves in a circular orbit, it is well known that the point L_4 is stable in the linear approximation. The Hamiltonian, in suitable canonical variables, can then be given the form (3.4), with frequencies $\omega_1 \simeq 0.99676$, $\omega_2 \simeq -0.080464$ and $\omega_3 = 1$. The fact that the frequencies have different signs is a major obstacle in proving that the point is stable also for the complete system. The first rigorous result was actually produced in the framework of KAM theory, as illustrated in the introduction. The natural question is whether a Nekhoroshev like result can be found for the spatial case, where the KAM theory cannot exclude the possibility of the Arnold diffusion.

A direct application of theorem 4.2 to this case presents two problems: i) the Hamiltonian is not even in the momenta; ii) we cannot guarantee that the frequencies are nonresonant. The former problem is in fact a minor one, since one can give an indirect proof of the consistency of the construction with the methods of the normal form theory. The latter problem appears instead to be more delicate, and needs a careful analysis. The relevant fact here is that at each step of the perturbation procedure the integer vectors $k \in \mathbf{Z}^n$ which can appear in the small denominators are a finite number, since at order s one has the limitations $0 < |k| \leq s + 2$ (recall that this fact has already been used in the condition (4.3) on the small denominators). Thus, since we are only interested in prime integrals truncated at order $r_{\rm opt}$, it is enough to avoid the resonances with $|k| \leq r_{\rm opt} + 2$. This means in practice that the frequencies are allowed to vary in a small interval, without seriously affecting the final result, so that the problem due to the fact that the resonances are dense in the frequency space turns out in fact to be less acute.

The stability of the point L_4 has been investigated, in the light of Nekhoroshev's like results, by applying the methods of the normal form theory (Giorgilli et al., 1989). As already explained, these methods are essentially equivalent to the direct construction of prime integrals, but offer the advantage of being easily applicable also to the resonant case.

The question is the following: consider a domain which, in the canonical variables which give the Hamiltonian the form (3.4), is a ball of radius ϱ centered at the point L_4; say that ϱ_0 is a radius of effective stability up to time T in case one can guarantee that all the orbits starting in the ball of radius ϱ_0 are confined inside a ball of radius $2\varrho_0$ up to time T. Then determine ϱ_0 in such a way that the time T if of the order of the estimated age of the universe.

The size of ϱ_0 can be explicitly estimated, thus obtaining the size of the ball of effective stability; if then one projects such a domain on the plane of the Jupiter's orbit one gets a region which has roughly an elliptic shape, with axes of about 1 to 10 kilometers. This is not a fully realistic result, but is very promising.

A better estimate can be obtained by explicitly performing some perturbative steps, and then applying analytical estimates. What one does essentially is to reproduce all the considerations of the previous sections, but replacing the harmonic actions I_l with $\Phi^{(l,r)}$, namely an integral truncated to an arbitrary order r (usually not the optimal one). This gives a substantial improvement, since the size of the estimated domain of effective stability turns out to be about 10^6 kilometers (Simó, 1989). Although there is no proof that this is an optimal result, it is surely a realistic one.

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