Notes on Non-Linear Hamiltonian Dynamics (in progress)

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Contents

Preface

This preliminary notes include some selected topics on non-linear dynamics, chaos and diffusion in Hamiltonian systems presented in a pedagogical way following the transition from integrable to non-integrable systems. The issue of resonance interactions is thoroughly discussed leading naturally to the very origin of chaos.

The structure of the notes follows Chirikov's formulation given in his former outstanding review "A Universal Instability of Many-Dimensional Oscillatory Systems" (1979), which was primarily devoted to the understanding of the so-called Arnol'd diffusion. In fact, the description given therein enables to introduce the non-math students and researchers to this stiff subject in a fairly comprehensive fashion.

Further, we will discuss the role that Arnol'd diffusion might play in actual world and address a more difficult problem, the global diffusion in phase space, focusing in applications to astronomical and astrophysical problems. However, all these exciting subjects are still in progress and have not been written yet.

This preliminary version includes discussions concerning the pendulum model, averaging, a model for a non-linear resonance, the standard and whisker mappings, the stochastic layer, chaotic motion and just an application of several of these tools to the orbital family in axisymmetric elliptical or disk galaxies.

This notes come out from a course delivered at the National University of La Plata, since 2000. Therefore, I owe a debt of gratitude to all the alert audience, who helped me with many useful suggestions.

This compendium will definitely appeal not only to both advanced undergraduate and graduate students in physics, astronomy, astrophysics, astrodynamics, chemistry, etc., but also to researchers, specially those mainly interested in problems such as the stability of Hamiltonian systems, chaos

and phase space diffusion.

A minimum background in Hamiltonian dynamics is assumed, in particular canonical transformations and action-angle formulation are extensively used along this notes. For instance, any text book on Classical Mechanics is enough to follow this non-rigorous notes on non-linear Hamiltonian dynamics. Anyway, the first chapter contains a review on basic topics of the Hamiltonian formulation of mechanics written by Prof. Héctor Vucetich.

As it was already mentioned, all subjects presented here follow the heuristic but clear approach of Chirikov (1979). However, some other references should be mentioned like Cincotta and Simó (2000), Cincotta (2002), Arnold (1989) among others. In the revised and enlarged version, new references should definitively be addressed.

I would like to acknowledge to my colleagues Roberto Barrio (Department of Applied Mathematics of University of Zaragoza), Ivan Shevchenko (Pulkovo Observatory of the Russian Academy of Sciences) among others, for useful suggestions and comment that allow me to improve the presentation of chapters 7, 8 and 9.

Chapter 1

Review of Hamiltonian Dynamics

This chapter was entirely written by Prof. Héctor Vucetich

The Lagrangian formulation of Mechanics is invariant under general coordinate transformations, namely, the Lagrange equations are invariant under these transformations and suggest the use of generalized coordinates $\{q_i\}$. This generality is, however, not enough to cover many important problems in mechanics.

Generalized momenta $\{p_i\}$ play an important role in mechanics, as variables closely associated to the dynamics of a given system. For instance, for every symmetry of a given system, there is a conserved generalized momentum p_i.

The two remarks above suggest a new formulation of mechanics where coordinates and momenta play a symmetric part: the *Hamiltonian formulation.*

1.1 A Reminder of the Lagrangian formulation

Let us introduce the Lagrangian formulation through the variational principle of Hamilton. This is one of the most powerful tools of theoretical physics, worth to be recalled.

1.1.1 Lagrangian dynamical systems

Let us introduce this notions through a simple example: a conservative mechanical system.

Let the *configuration space* of a mechanical system be a manifold parameterized with a set of n *generalized coordinates* $\{q_i\}$, describing the n *degrees of freedom* of the system. The time derivatives of these generalized coordinates are called the generalized velocities $\{\dot{q}_i\}$. These two sets describe the *kinematical properties* of the mechanical system. For the sake of simplicity, we shall often denote these two sets in the short form $(q, \dot{q})$.

Then, the *kinetic energy* T of our system will have the form

$$T = \frac{1}{2} \sum_{ik} g_{ik}(q,t) \dot{q}_i \dot{q}_k \tag{1.1}$$

where the functions g_{ik} are proportional to the masses of the involved bodies.

Since we shall work with *conservative systems*, there exists a *potential energy* represented by a smooth function

$$V = V(q,t) \tag{1.2}$$

The *Lagrangian function* (or simply, the *Lagrangian*) of the mechanical system is defined as

$$L(q, \dot{q}, t) = T - V \tag{1.3}$$

More general dynamical systems may have more general expressions for the Lagrangian. For instance, the kinetic energy may have a general form

$$T = B(q,t) + \sum_i A_i(q,t) \dot{q}_i + \frac{1}{2} \sum_{ik} g_{ik}(q,t) \dot{q}_i \dot{q}_k \tag{1.4}$$

and there may exist a *generalized potential* of the form

$$U = U(q, \dot{q}_i, t) \tag{1.5}$$

In summary, a *general Lagrangian dynamical system* is characterized by

1. the existence of a kinematical space: a differential manifold described by a set of generalized coordinates q and the corresponding generalized velocities $\dot{q}$,

2. the existence of a Lagrangian of the form

$$L = L(q, \dot{q}, t) \tag{1.6}$$

smooth enough over the kinematical space.

The dynamical properties of the system are derived from the Hamilton principle.

1.1.2 Least action principle

The *Hamilton principle* (also called the *least action principle*) states that the physical trajectory of the system minimizes the *action functional*

$$S = \int_{t_{\rm i}, q_{\rm i}}^{t_{\rm f}, q_{\rm f}} L(q, \dot{q}, t) dt \tag{1.7}$$

namely

$$\delta S = \delta \int_{t_i, q_i}^{t_f, q_f} L(q, \dot{q}, t) dt = 0 \tag{1.8}$$

where the symbol δ denotes a variation of the trajectory around the physical one.

The notation in equations (1.7) and (1.8) emphasizes that the initial and final coordinates q_i and q_f are prescribed at the initial and final times t_i and t_f respectively.

The problem defined by the above equations is an standard problem of the calculus of variations, and the minimum condition are given by the *Euler-Lagrange equations*

$$\frac{d}{d\,t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{1.9}$$

Let us make some remarks on the Lagrangian formulation.

1. From the above proof, a change of variable of the form

$$q_i' = q_i'(q_1, \ldots, q_n)$$

does not change the form of the Lagrange equations (1.9).

2. If the generalized momenta are introduced through the definition

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.10}$$

and the generalized forces through

$$Q_i = \frac{\partial L}{\partial q_i} \tag{1.11}$$

the Lagrange equation take the Newton-like form

$$\dot{p}_i = Q_i \tag{1.12}$$

3. The Lagrangian $L(q, \dot{q}, t)$ of a given dynamical system is not unique: another Lagrangian

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d\,F}{d\,t} \tag{1.13}$$

where F is an arbitrary function, yields the same equations of motion.

1.1.3 Examples

Let us consider a few examples of Lagrangian systems.

The harmonic oscillator

This is the simplest example of a Lagrangian system. If we write

$$T = \frac{1}{2} m \dot{q}^2 \qquad\qquad V = \frac{1}{2} m \omega^2 q^2 \tag{1.14}$$

the Lagrangian takes the form

$$L = T - V = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \tag{1.15}$$

and the resulting equation of motion is the elementary one

$$\ddot{q} + \omega^2 q = 0 \tag{1.16}$$

Particle moving on a curve

Consider a bead moving over a wire twisted along a curve described in parametric form

$$\mathbf{r} = \mathbf{r}(\xi) \tag{1.17}$$

We shall use ξ as a generalized coordinate. Then the velocity of the bead along the curve will be

$$\mathbf{v} = \frac{d\,\mathbf{r}}{d\,\xi}\dot{\xi} \tag{1.18}$$

and its kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{d\,\mathbf{r}}{d\,\xi}\right)^2\dot{\xi}^2 = \frac{1}{2}mF(\xi)\dot{\xi}^2 \tag{1.19}$$

where $F(\xi)$ is the squared modulus of the tangent vector.

If the particle moves under a force field with a potential $V(\mathbf{r})$, the Lagrangian of the system is

$$L = \frac{1}{2}mF(\xi)\dot{\xi}^2 - V[\mathbf{r}(\xi)] \tag{1.20}$$

Motion in a central field of force

Motion of a particle in a central field of force is better treated in polar coordinates. The kinetic energy is easily deduced from the length of arc in polar coordinates, and so the Lagrangian of the system is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{1.21}$$

The equations of motion result immediately from the above Lagrangian. We find first the generalized momenta

$$p_r = m\dot{r}, \qquad p_\phi = mr^2\dot{\phi}, \tag{1.22}$$

and the corresponding generalized forces

$$Q_r = -\frac{d\,V}{d\,r} + mr\dot{\phi}^2, \qquad Q_\phi = 0. \tag{1.23}$$

The radial generalized force is composed by the central force together with the centrifugal force

$$mr^2\dot{\phi} = \frac{mv_\phi^2}{r}$$

and the tangential force is zero, since the coordinate ϕ is absent in the Lagrangian.

A coordinate that only appears in the Lagrangian through its time derivative is called a *cyclic coordinate* (and often a *cyclic variable* too): the generalized force associated with a cyclic coordinate is zero and the corresponding generalized momentum is a constant of motion.

1.1.4 Symmetry and conservation laws

Consider a group $\mathcal{G}$ of transformations, depending on N_g parameters, acting over the configuration space of a dynamical system. The group elements g^r will be represented as analytic changes of coordinates of the form

$$q_i' = g_i^r(q) \tag{1.24}$$

The group $\mathcal{G}$ is a *symmetry* of a dynamical Lagrangian system if the transformed action of the system satisfies

$$L'(q', \dot{q'}) = L(q, \dot{q}) + \dot{F} \tag{1.25a}$$

or

$$S' = S + \Delta F \tag{1.25b}$$

The connection between symmetry and conservation laws is contained in the above definition. Consider an uniparametric subgroup of $\mathcal{G}$ and let α be the parameter. An infinitesimal transformation will change each coordinate in the form

$$\delta q_i = \delta\alpha \left.\frac{\partial q_i}{\partial \alpha}\right|_{\alpha=0} \tag{1.26}$$

The change in the action induced by the transformation (1.26) is

$$\delta S = \delta\alpha \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha} \right) dt \tag{1.27}$$

which after integration by parts of the last term can be recast in the form

$$\delta S = \delta\alpha \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\bigg|_{t_\mathrm{i}}^{t_\mathrm{f}} \tag{1.28}$$

If $q_i(t)$ represents the physical path of the system, the first term in (1.28) is zero, and the condition of invariance of the action $\delta S = 0$ yields

$$\sum p_i \frac{\partial q_i}{\partial \alpha}\bigg|_{t=t_\mathrm{i}} = \sum p_i \frac{\partial q_i}{\partial \alpha}\bigg|_{t=t_\mathrm{f}} \tag{1.29}$$

which states a conservation law.

If α is chosen as a generalized coordinate, the above conservation law takes a very simple form:

$$p_\alpha(t_\mathrm{i}) = p_\alpha(t_\mathrm{f}) \tag{1.30}$$

which states the conservation of the associated momentum. Of course, α will be a cyclic variable in this case.

Time invariance and energy conservation

A system whose Lagrangian does not depend explicitly on time is called *time translation invariant* system. This invariance is not covered by the simple version of Noether's theorem. If the system is time translated by the infinitesimal amount δt, its action S changes in

$$\delta S = \delta t \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) dt - \delta t[L(t_\mathrm{f}) - L(t_\mathrm{i})] \tag{1.31}$$

With the help of the equation of motion (1.9), the first term can be written in the form

$$\delta t \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \right) dt = \delta t \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i \frac{d}{dt}(p_i \dot{q}_i) dt$$

and we finally obtain the conservation law

$$\left(\sum_i p_i \dot{q}_i - L \right)\bigg|_{t=t_\mathrm{i}} = \left(\sum_i p_i \dot{q}_i - L \right)\bigg|_{t_\mathrm{f}} \tag{1.32}$$

The quantity

$$H = \sum_i p_i \dot{q}_i - L \tag{1.33}$$

is called the *Hamiltonian* function of the system. It is really a function of (q_i, p_i) rather than $(q_i, \dot{q}_i)$ since (1.33) is minus a Legendre transformation of the Lagrangian (Sec. 1.2). For a standard Lagrangian

$$L = \frac{1}{2} \sum_{ij} g_{ij} \dot{q}_i \dot{q}_j - V(q)$$

whose generalized momenta are

$$p_i = \sum_j g_{ij} \dot{q}_j$$

the Hamiltonian is the total energy of the system

$$H = \sum_i p_i \dot{q}_i - L = \frac{1}{2} \sum_{ij} g_{ij} \dot{q}_i \dot{q}_j + V(q) = E \tag{1.34}$$

Thus, energy conservation is a consequence of time translation invariance. This nice result is a consequence of another restriction: Lagrangian systems do not admit dissipative forces since a Lagrangian cannot be defined for them.

1.2 Introduction to the Hamiltonian formulation

1.2.1 The Hamilton equations

In order to introduce coordinates and momenta in a symmetrical way, let us perform a *Legendre transformation* of the Lagrangian

$$H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t) \tag{1.35}$$

which transforms the Lagrangian L, a function of coordinates and velocities into a new function, the *Hamiltonian function*, H of the coordinates and

momenta[1]. This is easy to check by computing the total differential of both functions. We have

$$dL(q,\dot{q},t) = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} \tag{1.36}$$

while

$$\begin{aligned} dH(q,p,t) &= \sum_i \left(p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i \right) - \frac{\partial L}{\partial t} \\ &= \sum_i \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i \right) - \frac{\partial L}{\partial t} \\ &= \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \end{aligned} \tag{1.37}$$

Comparison of the last two lines in equation (1.37) and the Lagrange equations (1.9) lead us to the *Hamilton equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.38a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.38b}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{1.38c}$$

These beautiful symmetrical equations are also called *canonical equations*. They describe the mechanical system and are equivalent to the Lagrange equations if velocities and momenta can be exchanged, namely, if the transformation defined by equation (1.10) can be inverted.

A variational principle

The Hamilton equations can be derived from a variational principle. Substitution of the Lagrangian by its expression in terms of the Hamiltonian leads to

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_i p_i \dot{q}_i - H(q,p) \tag{1.39}$$

The variation must be done on both (q,p), with fixed endpoint condition for the q but no conditions on the p.

[1]Really, equation (1.35) is minus the Legendre transformation of the Lagrangian.

1.2.2 Examples

Let us consider the same examples we saw in Section 1.1.3.

Particle moving on a curve

Equation (1.20) is the corresponding Lagrangian. The generalized momentum is

$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = mF(\xi)\dot{\xi} \tag{1.40}$$

and the Hamiltonian

$$\begin{aligned} H(\xi, p_\xi) &= p_\xi \dot{\xi} - L(\xi, \dot{\xi}) \\ &= \frac{p_\xi^2}{2mF(\xi)} + V[\mathbf{r}(\xi)] \end{aligned} \tag{1.41}$$

The corresponding Hamilton equations are

$$\dot{\xi} = \frac{\partial H}{\partial p_\xi} = \frac{p_\xi}{mF(\xi)} \tag{1.42a}$$

$$\dot{p_\xi} = -\frac{\partial H}{\partial \xi} = -\frac{p_\xi^2}{2mF(\xi)}\frac{F'(\xi)}{F(\xi)} - \nabla V \cdot \mathbf{r}' \tag{1.42b}$$

Motion in a central field of force

The corresponding Lagrangian is written in (1.21), from which we derive the generalized momenta

$$p_r = m\dot{r} \qquad\qquad p_\phi = mr^2\dot{\phi} \tag{1.43}$$

and the corresponding Hamiltonian

$$\begin{aligned} H &= p_r\dot{r} + p_\phi\dot{\phi} - L \\ &= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r) \end{aligned} \tag{1.44}$$

and the canonical equations are

$$\dot{r} = \frac{p_r}{m} \qquad\qquad \dot{\phi} = \frac{p_\phi}{mr^2} \tag{1.45a}$$

$$\dot{p_r} = -V'(r) - \frac{p_\phi^2}{mr^3} \qquad\qquad \dot{p_\phi} = 0 \tag{1.45b}$$

The last equation shows that p_ϕ is a constant of motion, namely, angular momentum

$$p_\phi = L = mr^2\dot{\phi} \tag{1.46}$$

This is, of course, a consequence of isotropy around the origin.

1.2.3 Phase space

Le us introduce a compact notation for the Hamiltonian formulation of mechanics. The symmetry between coordinates and momenta suggest us to introduce a *state vector*, of dimension $2n$, whose components are

$$\boldsymbol{\xi} = \begin{pmatrix} q \\ p \end{pmatrix} \tag{1.47}$$

and a *fundamental matrix*

$$\mathbb{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.48}$$

whose elements are $n \times n$ matrices.

With this notation, the Hamilton equations take the very compact form

$$\dot{\boldsymbol{\xi}} = \mathbb{J}\frac{\partial H}{\partial \boldsymbol{\xi}} \tag{1.49}$$

This latter equation suggest an interesting geometrical interpretation of the motion of the system. The vector $\boldsymbol{\xi}$ describes a curve in a $2n$-dimensional space whose points represent possible states of the dynamical system. We call this space the *phase space* of the system.

As an example, consider the phase space for a harmonic oscillator whose Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{1.50}$$

The corresponding phase space is two-dimensional, but energy conservation restricts the motion of the system to the *energy surface* (in this case a closed curve)

$$H = E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{1.51}$$

i.e. a ellipse with semi-axes

$$a = \sqrt{\frac{2E}{m\omega^2}} \qquad\qquad b = \sqrt{2mE}$$

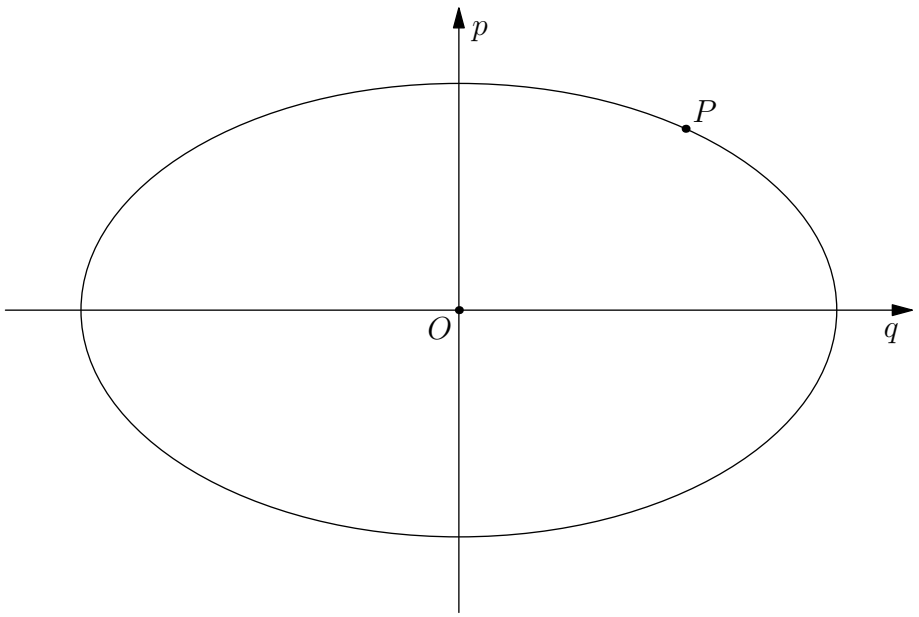

Figure 1.1: Phase space of the harmonic oscillator.

The point representing the state of the system moves along the ellipse with a constant angular velocity ω:

$$x = a\sin(\omega t + \phi) \qquad\qquad p = b\cos(\omega t + \phi)$$

For systems with more degrees of freedom, the phase space is not easily represented. However, for all systems the motion of the representative point is restricted to the energy hyper-surface defined by

$$H(p, q) = E. \tag{1.52}$$

Additional restraints will be introduced by other conservation laws such as angular momentum conservation. These restrictions allow sometimes a visual representation of the $2n$-dimensional phase space.

For instance, motion in a central field of force leads to a four-dimensional phase space. Angular momentum conservation, however, restricts the path of the representative point to the three dimensional subspace $p_\phi = L$ and energy conservation again limits it to the two dimensional surface

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r) = E \tag{1.53}$$

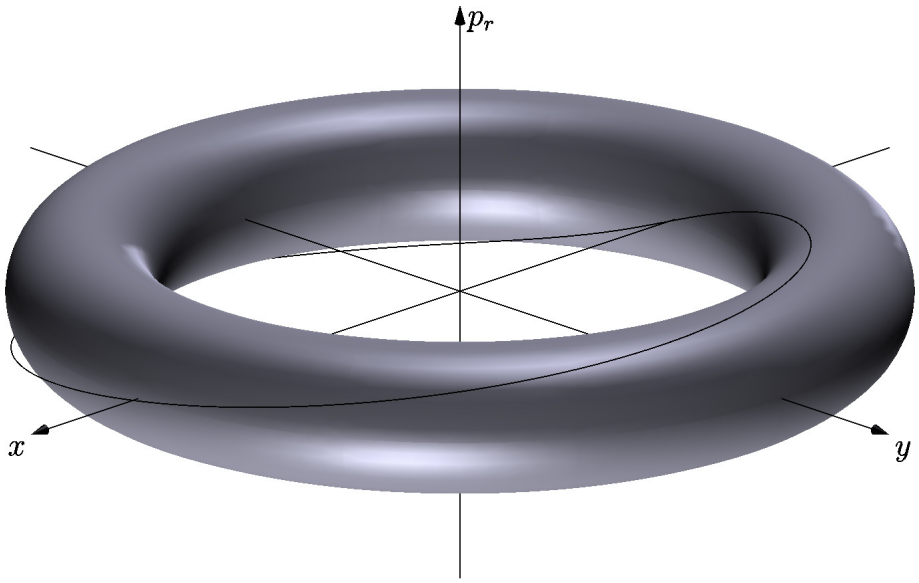

Figure 1.2: Illustrative phase space for motion in a central field.

For the particular case of bounded motion, this surface is a torus embedded in the subspace $p_\phi = L$.

1.3 Canonical Transformations

The Lagrangian formulation of mechanics is characterized by the covariance of the equations of motion under general transformations of coordinates: the Euler-Lagrange equation have the same form in any system of coordinates. In a similar way, the canonical equation are invariant under a set of general transformations in phase space, called *canonical transformations*.

Canonical transformation are defined as a set of coordinate transformations

$$Q_i = Q_i(q, p, t) \qquad\qquad P_i = P_i(q, p, t) \tag{1.54}$$

such that the canonical equations (1.38) keep their form in the new variables

$$\dot{Q}_i = \frac{\partial K(P, Q, t)}{\partial P_i} \qquad\qquad \dot{P}_i = -\frac{\partial K(P, Q, t)}{\partial Q_i} \tag{1.55}$$

where $K(P, Q, t)$ denotes the new Hamiltonian.

The simplest characterization of canonical transformations is obtained from the variational principle considered in Sec. 1.2.1. The Hamilton equations will be form invariant if the corresponding Lagrangian differ only in a total time derivative

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{d\,F_1}{d\,t} \tag{1.56}$$

where F_1 is called the *generating function* of the canonical transformation. Writing (1.56) in the form

$$dF_1 = \sum_i p_i dq_i - \sum_i P_i dQ_i + (K - H)dt \tag{1.57}$$

is clear that F_1 is a function of the coordinates q and Q, and that the momenta and the Hamiltonian difference are obtained as

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad K(Q,P,t) = H(q,p,t) + \frac{\partial F_1}{\partial t} \tag{1.58}$$

With the use of Legendre transformations other forms of the generating function can be obtained. For instance, defining

$$F_2(q,P,t) = F_1(q,Q,t) + \sum_i Q_i P_i \tag{1.59}$$

since

$$\begin{aligned} dF_2 &= dF_1 + \sum_i (Q_i dP_i + P_i dQ_i) \\ &= \sum_i (p_i dq_i + Q_i dP_i) + (K - H)dt \end{aligned} \tag{1.60}$$

and so F_2 is a function of the old variables and new momenta, from which the remaining canonical variables are found in the form

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = H + \frac{\partial F_2}{\partial t} \tag{1.61}$$

With suitable Legendre transformations, it is possible to recast the generating function in two other standard forms $F_3(p,Q,t)$ and $F_4(p,P,t)$.

However, many of the canonical transformations in practical applications are used to change a few pairs of the canonical variables, leaving all others unchanged.

1.3.1 Examples

Let us present a few simple examples of canonical transformations.

Elementary canonical transformations

The generating function

$$F_2 = \sum_i q_i P_i \tag{1.62}$$

is the *unit canonical transformation*, that keeps coordinates and momenta.

Another elementary canonical transformation that exchanges coordinates and momenta is

$$F_1 = \sum_i q_i Q_i \qquad P_i = q_i \qquad Q_i = -p_i \tag{1.63}$$

A third example of generating function is

$$F_2 = \sum_i f_i(q) P_i \tag{1.64}$$

which generates coordinate transformations (also called *point transformations*). So, canonical transformations are generalizations of coordinate transformations.

Application to the harmonic oscillator

Let us write the Hamiltonian of the harmonic oscillator in the form

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{1.65}$$

Since the Hamiltonian is a quadratic form in the coordinates and momenta, let us seek a canonical transformation of the form

$$q = \sqrt{\frac{2}{m\omega^2}} f(P) \cos Q \qquad p = -\sqrt{2m} f(P) \sin Q \tag{1.66}$$

where $f(P)$ is an unknown function. Substitution of (1.66) into (1.65) yields the new Hamiltonian

$$K = [f(P)]^2 \tag{1.67}$$

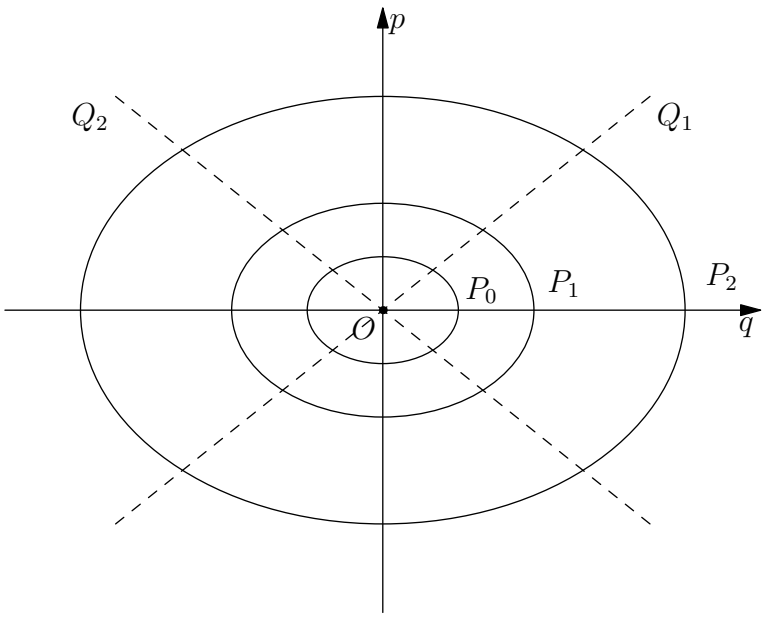

Figure 1.3: A canonical mapping for the harmonic oscillator.

thus, if (1.66) represents a canonical transformation, Q is a cyclic variable and P is a conserved quantity.

Elimination of $f(P)$ between the equations (1.66) yield the condition

$$p = -m\omega q \tan Q = \frac{\partial F_1}{\partial q} \tag{1.68}$$

and therefore

$$F_1(q, Q) = -\frac{1}{2} m\omega q^2 \tan Q + g(Q) \tag{1.69}$$

where $g(Q)$ is an arbitrary function. With $g(Q) = 0$, we obtain

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2}\frac{m\omega q^2}{\cos^2 Q} = \frac{1}{2m\omega}(p^2 + m^2\omega^2 q^2) \tag{1.70}$$

and our final result

$$K = \omega P = H(P) \tag{1.71}$$

Figure 1.3 shows the effect of the mapping: constant energy curves are mapped on P momentum while constant phase curves are mapped on Q

coordinate. As a result of the mapping, the Q variable is cyclic and from the Hamilton equations we obtain

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \quad \Longrightarrow \quad Q = \omega t + \phi$$

The canonical transformation, found with simple mathematical tools, has completely solved a the simplest nontrivial problem in mechanics.

1.3.2 Symplectic structure

Let us examine the invariant structures in phase space under canonical transformations. Using the notation of equations (1.47)–(1.48) the transformation law can be written, for time independent transformations, in the form

$$\boldsymbol{\eta} = \boldsymbol{F}(\boldsymbol{\xi}) \tag{1.72}$$

Under this transformation the Hamilton equations should behave as

$$\dot{\boldsymbol{\xi}} = \mathbb{J}\frac{\partial H}{\partial \boldsymbol{\xi}} \quad \rightarrow \quad \dot{\boldsymbol{\eta}} = \mathbb{J}\frac{\partial K}{\partial \boldsymbol{\eta}} \tag{1.73}$$

where K is the new Hamiltonian.

Let M be the Jacobian matrix of the transformation. Then

$$\dot{\boldsymbol{\eta}} = \mathsf{M}\dot{\boldsymbol{\xi}} = \mathsf{M}\mathbb{J}\frac{\partial H}{\partial \boldsymbol{\xi}} = \mathsf{M}\mathbb{J}\mathsf{M}^{\top}\frac{\partial K}{\partial \boldsymbol{\eta}} \tag{1.74}$$

Comparison with (1.73) yields the result

$$\mathbb{J} = \mathsf{M}\mathbb{J}\mathsf{M}^{\top} \tag{1.75}$$

i.e. the unit symplectic matrix is invariant under canonical transformations.

The geometry having $\mathbb{J}$ as an invariant is called *symplectic geometry*. The characteristic group of transformations is the *symplectic group* of matrices satisfying (1.75). Another symplectic invariant built with $\mathbb{J}$ is the *symplectic product*

$$[\boldsymbol{\xi}, \boldsymbol{\eta}] = \boldsymbol{\xi}\mathbb{J}\boldsymbol{\eta} \tag{1.76}$$

1.3.3 Integral invariants

One of the most interesting results of the Hamiltonian formulation is the existence of *integral invariants*: geometrical structures invariant under canonical transformations.

The simplest integral invariant can be found from (1.56) integrating over a closed curve C in phase space

$$\oint_C \sum_i p_i dq_i = \oint_{\hat{C}} \sum_i P_i dQ_i \tag{1.77}$$

where $\hat{C}$ is the image of C under the canonical transformation. The quantity

$$I_R^1(C) = \oint_C \sum_i p_i dq_i \tag{1.78}$$

is called a *relative integral invariant*. This can be written as

$$I_R^1(C) = \frac{1}{2}\oint_C (p_i dq_i - q_i dp_i) = \frac{1}{2}\oint_C [\boldsymbol{\xi}, d\boldsymbol{\xi}] \tag{1.79}$$

which explicitly shows the symplectic structure of the invariant.

Applying the Stokes theorem to the line integrals in (1.77) we obtain

$$\iint_S \sum_i dp_i dq_i = \iint_{\hat{S}} \sum_i dP_i dQ_i \tag{1.80}$$

where S is the surface in phase space limited by C and $\hat{S}$ is its image. The integral

$$I^1(S) = \iint_S \sum_i dp_i dq_i \tag{1.81}$$

is called an *absolute integral invariant* whose symplectic structure is

$$I^1(S) = \iint_S [d\boldsymbol{p}_i, d\boldsymbol{q}_i] \tag{1.82}$$

In a similar way, there exists a sequence of invariants

$$I^2 = \iiiint_{V_4} \sum_{i<j} dq_i dp_i dq_j dp_j \qquad \cdots \qquad I^n = \int \cdots \int_{V_n} \prod_i dq_i dp_i \tag{1.83}$$

each of them a volume of a k-dimensional set in phase space. These invariants play an important role in the foundations of statistical mechanics.

1.4 Poisson Brackets

The composition of canonical transformation is another canonical transformation and this suggests the possibility of studying *infinitesimal canonical transformations* whose composition generate finite canonical transformations. In turn, infinitesimal canonical transformations reveal another geometrical structure in phase space, namely, *Poisson brackets*.

1.4.1 Infinitesimal canonical transformations

An infinitesimal canonical transformation is generated by a generating function close to the identity

$$F_2 = \sum_i q_i P_1 + \epsilon G(q, P) \tag{1.84}$$

where ϵ is an infinitesimal parameter and $G(q, P)$ is the *infinitesimal generator* of the transformation.

Using (1.61) one finds, neglecting higher orders in ϵ

$$Q_i = q_i + \epsilon \frac{\partial G(q,p)}{\partial q_i} \qquad\qquad P_i = p_i - \epsilon \frac{\partial G(q,p)}{\partial p_i} \tag{1.85}$$

that can be recast in symplectic form

$$\boldsymbol{\Xi} = \boldsymbol{xi} + \epsilon \mathbb{J} \frac{\partial G}{\partial \boldsymbol{\xi}} \tag{1.86}$$

or, in differential equation form

$$\frac{d\,\boldsymbol{\xi}}{d\,\lambda} = \mathbb{J} \frac{\partial G}{\partial \boldsymbol{\xi}} \tag{1.87}$$

with the obvious replacement $\epsilon \to d\lambda$.

Comparison of (1.87) with the Hamilton equations (1.49) we deduce that the Hamiltonian is the infinitesimal generator of a canonical transformation that describes the time evolution of the Hamiltonian system. This beautiful result suggest a new approach to the integration of the equation of motion: find a canonical transformation that "freezes" the motion of the system.

1.4.2 Poisson brackets

Let $F(q,p,t)$ be a function representing some physical quantity. Common examples are the Hamiltonian $H(q,p)$ or the z-component of angular momentum $L_z = xp_y - yp_x$. Consider now the action of an infinitesimal canonical transformation (1.84) on $F(q,p)$. A series expansion around the original point in phase space yields

$$F(Q,P) = F(q,p) + \sum_i \left(\frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_i}\delta p_i \right)$$

and using (1.85) we obtain

$$F(Q,P) = F(q,p) + \epsilon \{F,G\} \tag{1.88}$$

where we have introduced the *Poisson bracket* of functions F and G

$$\{F,G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right) \tag{1.89}$$

or, in symplectic notation

$$\{F,G\} = \frac{\partial F}{\partial \boldsymbol{\xi}}^{\top} \mathbb{J} \frac{\partial G}{\partial \boldsymbol{\xi}} \tag{1.90}$$

The Hamilton equations can be written in Poisson bracket form

$$\dot{q}_i = \{q_i, H\} \qquad \dot{p}_i = \{p_i, H\} \qquad \text{or} \qquad \dot{\boldsymbol{\xi}} = \{\boldsymbol{\xi}, H\} \tag{1.91}$$

Using (1.88) we can easily write

$$\frac{d\,F}{d\,t} = \{F,H\} + \frac{\partial F}{\partial t} \tag{1.92}$$

and for a conserved physical quantity

$$\{F,H\} = 0 \tag{1.93}$$

which is an extremely concise expression for a conservation law.

1.4.3 Properties of Poisson brackets

Poisson brackets satisfy several useful identities.

$$\{F, G\} = -\{G, F\} \qquad \text{Antisymmetry} \qquad (1.94a)$$
$$\{aF + bG, H\} = a\{F, H\} + b\{G, H\} \qquad \text{Linearity} \qquad (1.94b)$$
$$\{FG, H\} = \{F, H\} G + F\{G, H\} \qquad \text{Leibniz Property} \qquad (1.94c)$$

These equations show that Poisson Brackets are differentiation operators on phase space.

Another important property is the *Jacobi identity*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \qquad (1.95)$$

whose proof is straightforward but rather long.

The *fundamental Poisson brackets* are

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \qquad \{q_i, p_j\} = \delta_{ij} \qquad \text{or} \qquad \{\boldsymbol{\xi}, \boldsymbol{\xi}\} = \mathbb{J} \qquad (1.96)$$

Last but not least, Poisson brackets are canonical invariants. Consider the Poisson brackets computed with a particular set of canonical variables $\boldsymbol{\xi} = (q, p)$

$$\{F, G\}_{\boldsymbol{\xi}} = \frac{\partial F}{\partial \boldsymbol{\xi}}^{\top} \mathbb{J} \frac{\partial G}{\partial \boldsymbol{\xi}} \qquad (1.97)$$

Under a canonical transformation $\xi \to \eta$

$$\frac{\partial F}{\partial \boldsymbol{\eta}} = \mathsf{M}^{\top} \frac{\partial F}{\partial \boldsymbol{\xi}} \qquad \frac{\partial G}{\partial \boldsymbol{\eta}} = \mathsf{M}^{\top} \frac{\partial G}{\partial \boldsymbol{\xi}} \qquad (1.98)$$

and so

$$\{F, G\}_{\boldsymbol{\eta}} = \frac{\partial F}{\partial \boldsymbol{\xi}}^{\top} \mathsf{M}\mathbb{J}\mathsf{M}^{\top} \frac{\partial G}{\partial \boldsymbol{\xi}} = \{F, G\}_{\boldsymbol{\xi}} \qquad (1.99)$$

which proves the invariance.

1.4.4 Liouville's theorem

Since the motion of a Hamiltonian system can be described as the unfolding of a canonical transformation, the integral invariants (Sect. 1.3.3) are preserved through the time evolution. Thus, if S_0 is a two dimensional region at time

$t = t_0$, its image under the motion at $t = t_1$ will be S_1 and they will have the same area

$$I^1(S_0) = \iint_{S_0} \sum_i dp_i dq_i = \iint_{S_1} \sum_i dp_i dq_i = I^1(S_1) \tag{1.100}$$

Similar theorems are valid for all invariants and, we obtain *Liouville's theorem*: volumes in phase space are preserved by the equations of motion.

It should be stressed that these invariants are not constants of motion: the latter are associated with each given solution of Hamilton equations while integral invariants are associated with set of states or sets of solutions.

A pictorial interpretation of integral invariants is to consider an *ensemble* of systems, sharing the same Hamiltonian, but each one with different initial conditions. Each member of the ensemble will evolve according to the Hamilton equations and the ensemble will describe a complex set in phase space. However, the volume of is section $I^k[V(t)_{2k}]$ will be time independent.

1.5 The Hamilton-Jacobi equation

Let us find a canonical transformation that "freezes" the motion of a Hamiltonian system with Hamiltonian $H(q, p, t)$. It will be enough to impose that the the transformed Hamiltonian K be independent of the canonical variables since then

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \qquad\qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \tag{1.101}$$

The simplest choice is $K = 0$.

Call $F_2 = S(q, P, T)$ the generating function and from the transformation equations (1.61) we find the condition

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{1.102}$$

which is the *Hamilton-Jacobi equation*: a nonlinear partial differential equation for S in $n+1$ variables, whose solution is the generating function of the canonical transformation.

We do not need the general solution of (1.102), which depends on an arbitrary function, but only a *complete solution* depending on $n+1$ constants α_i in the form

$$S = S(q_1, \cdots, q_n; \alpha_1, \cdots, \alpha_n) + \alpha_{n+1} \tag{1.103}$$

A solution of this form is called *Hamilton's principal function.* The additive constant α_{n+1} is irrelevant so we have n arbitrary constant that correspond to the new momenta. From the transformation equations we find

$$p_i = \frac{\partial S(q,\alpha,t)}{\partial q_i} \qquad\qquad Q_i = \frac{\partial S(q,\alpha,t)}{\partial \alpha_i} = \beta_i \tag{1.104}$$

This system of (generally nonlinear) equations may be inverted to obtain the original coordinates and momenta as a function of time

$$q_i = q_i(\alpha,\beta,t) \qquad\qquad p_i = p_i(\alpha,\beta,t) \tag{1.105}$$

Thus, the Hamilton-Jacobi equation furnishes very powerful tools to find the solution of a mechanical problem if a complete solution can be found.

The principal function of Hamilton has a very simple interpretation. Its total derivative is

$$\dot{S} = \sum_i p_i \dot{q}_i - H = L$$

and integrating

$$S = \int L dt + \alpha_{n+1} \tag{1.106}$$

which is the action computed along the physical trajectory of the system.

1.5.1 Examples

Let us analyze some important examples of application of the Hamilton-Jacobi equation.

The harmonic oscillator

This simple example of the Hamilton-Jacobi equation is very instructive. Since the Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \tag{1.107}$$

the Hamilton-Jacobi equation results

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 = -\frac{\partial S}{\partial t} \tag{1.108}$$

To find a complete solution we shall try the method of *separation of variables* in an additive form

$$S = W(q) + T(t)$$

which yields

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = -\frac{dT}{dt}$$

The first member of this equation is only function of q while the second member is function of t, so both must be equal to some constant α

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = \alpha \tag{1.109a}$$

$$\frac{dT}{dt} = -\alpha \tag{1.109b}$$

We finally get

$$S = -\alpha t + \sqrt{2m}\int \sqrt{\alpha - \frac{1}{2}m\omega^2 q^2}dq \tag{1.110}$$

and from this expression we find

$$\begin{aligned}\frac{\partial S}{\partial \alpha} = \beta &= -t + \sqrt{2m}\int \frac{dq}{2\sqrt{\alpha - \frac{1}{2}m\omega^2 q^2}} \\ &= -t + \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m\omega^2}{2\alpha}}q\right)\end{aligned} \tag{1.111}$$

and solving q we find the well known solution of the harmonic oscillator problem

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega(t+\beta) \tag{1.112}$$

Thus we can interpret the new momentum with the energy $\alpha = E$ and the new coordinate with minus the initial time $\beta = -t_0$ respectively.

In every conservative system a similar separation of time is possible, writing

$$S(q,\alpha) = -Et + W(q,\alpha) \tag{1.113}$$

where we have introduced the *Hamilton's characteristic function.* It satisfies the differential equation

$$H\left(q, \frac{\partial W}{\partial q}\right) = E \tag{1.114}$$

and generates a canonical transformation such that the new Hamiltonian is only a function of the new momentum $P_1 = E$ and the new coordinate is the (shifted) time

$$K = E(\alpha_i) \qquad\qquad Q_1 = t + \beta \tag{1.115}$$

Motion in a central field

This is another example that can be solved with the Hamilton-Jacobi equation in full three-dimensional space. The corresponding Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + V(r) \tag{1.116}$$

which leads to the corresponding Hamilton-Jacobi equation

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + V(r) + \frac{\partial S}{\partial t} = 0 \tag{1.117}$$

We can try to solve this problem with the method of variable separation, seeking a solution in the form

$$S = -\alpha_1 t + W(r, \theta, \phi) \qquad\qquad W = R(r) + \Theta(\theta) + \Phi(\phi) \tag{1.118}$$

which leads to the separation equations

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + V(r) = \alpha_1 = E \tag{1.119a}$$

$$\frac{d\,\Phi}{d\,\phi} = \alpha_3 = L_z \tag{1.119b}$$

$$\left(\frac{\partial \Theta}{\partial \theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = \alpha_2 = L^2 \tag{1.119c}$$

$$\left(\frac{\partial R}{\partial r}\right)^2 + \frac{L^2}{r^2} = 2m\alpha_1 = 2mE \tag{1.119d}$$

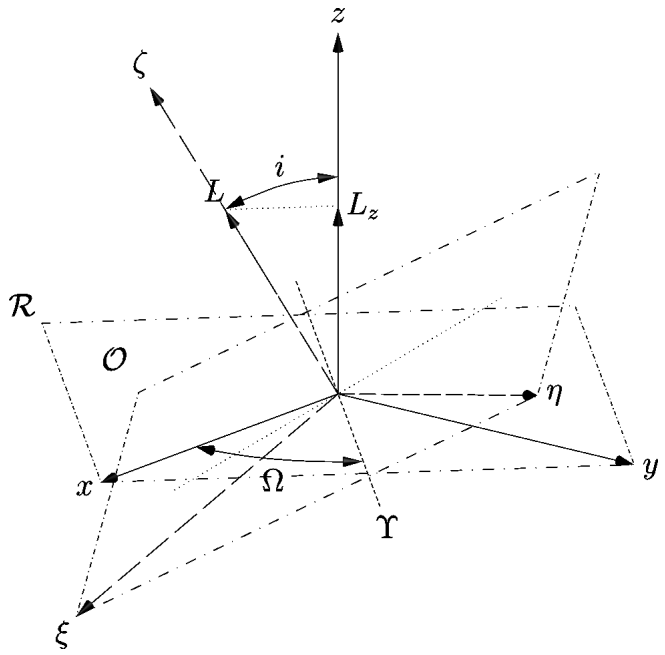

Figure 1.4: Orbital elements and integration constants.

where conventional names have been replaced for the α constants.

Thus Hamilton's principal function results in

$$\begin{aligned} S = & -Et - L_z\phi + \int \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\, d\theta \\ & + \int \sqrt{2mE - \frac{L^2}{r^2} - 2mV(r)}\, dr \end{aligned} \tag{1.120}$$

This is a complete solution of the Hamilton-Jacobi equation depending on the three constants E, L, L_z. Once the coordinate system is chosen, the geometry of the problem (*i.e.* the *orbital elements*) and its relation with the integration constants will be evident.

Let us choose a new coordinate system with axes (ξ, η, ζ) as shown in Figure 1.4. The (x, y) plane is the *reference plane* $\mathcal{R}$, while the new (ξ, η) plane is the *orbital plane* $\mathcal{O}$. The angle between both normals (the z and ζ axes) is the *inclination angle* of the orbit. The intersection of both planes is the *line of the nodes* Υ, forming an angle Ω with the x axis. The angular momentum vector $\mathbf{L}$ lies along the ζ axis and its projection on the z axis is L_z.

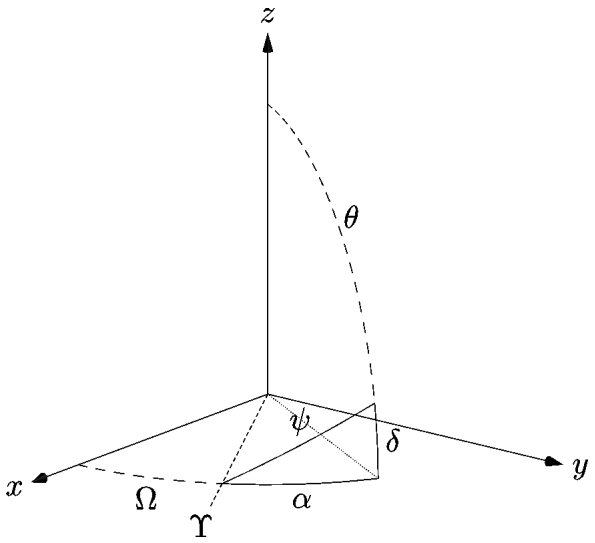

Figure 1.5: Relations between the angles.

With the above geometrical setting we can proceed to the integration of the central force problem. The second integral in (1.120) is straightforward but somewhat tedious.

$$\begin{aligned} I &= \int \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}d\theta \\ &= L\int \sqrt{\sin^2\theta - \cos^2 i}\frac{d\theta}{\sin\theta} \\ &= -L\int \sqrt{\sin^2 i - \cos^2\theta}\frac{d\cos\theta}{1-cos^2\theta} \\ &= -L\sin^2 i\int \frac{cos^2\psi d\psi}{1-\sin^2 i\sin^2\psi} \end{aligned} \tag{1.121}$$

In the first line the replacement $L_z = L\cos i$ was made and the last line was obtained with the change of variables

$$\cos\theta = \sin i\sin\psi \tag{1.122}$$

which can be interpreted geometrically from some well-known results in spherical trigonometry.

Let ψ be the angle the angle that the particle P forms with the node along the orbit plane $\mathcal{O}$ (Figure 1.5). Then in the spherical triangle of Figure 1.5

the relations between the angles are

$$\alpha = \phi - \Omega \tag{1.123a}$$

$$\delta = \frac{\pi}{2} - \theta \tag{1.123b}$$

$$\tan(\phi - \Omega) = \tan\psi \cos i \tag{1.123c}$$

$$\sin\delta = \sin i \sin\psi \tag{1.123d}$$

and the latter equation is equivalent to our variable change (1.122).

With the latter change of variables, the integral (1.121) can be easily computed

$$\begin{aligned} I &= -L\psi + L\cos i \arctan(\cos i \tan\psi) \\ &= -L\psi + L_z(\phi - \Omega) \end{aligned} \tag{1.124}$$

where we have used the result (1.123d) to simplify the result. The principal function of Hamilton (1.120) results after substitution

$$S = -Et - L_z\Omega - L\psi + \int \sqrt{2mE - \frac{L^2}{r^2} - 2mV(r)}\, dr \tag{1.125}$$

and the latter integrand can only be computed when the central potential $V(r)$ is known.

However, one can find the equations of motion from the above general expression for S. Since the conjugate canonical momentum of ψ is L

$$L = \frac{\partial S}{\partial \psi} \tag{1.126}$$

then we find the *equation of the orbit* computing

$$\begin{aligned} \beta_L = \psi_0 &= \frac{\partial S}{\partial L} \\ &= \psi - \int \frac{L}{\sqrt{2mE - \frac{L^2}{r^2} - V(r)}} \frac{dr}{r^2} \end{aligned} \tag{1.127}$$

The last equation of motion can be found from

$$\begin{aligned} \beta_E = t_0 &= \frac{\partial S}{\partial E} \\ &= -t + \int \frac{m dr}{\sqrt{2mE - \frac{L^2}{r^2} - V(r)}} \end{aligned} \tag{1.128}$$

which may be called the *time evolution equation*. Again, the integral can be computed only if $V(r)$ is given.

1.5.2 The Kepler problem

Let us consider the motion of a particle (which we shall denote as "planet" sometimes) in an inverse square force

$$V(r) = -\frac{k}{r} \tag{1.129}$$

where we shall limit ourselves to the $k > 0, E < 0$ case.

Although the last integrand in (1.125) can be explicitly computed, the result is far from illuminating and it is better to solve equations (1.127) and (1.128), which can be solved in terms of elementary functions.

The equation of the orbit in the Kepler problem is

$$\begin{aligned} \psi - \psi_0 &= \int \frac{L dr}{r^2 \sqrt{2mE - \frac{L^2}{r^2} + \frac{k}{r}}} \\ &= \arccos \frac{\frac{L}{r} - \frac{mk}{L}}{\sqrt{2mE + \frac{(mk)^2}{L^2}}} \end{aligned} \tag{1.130}$$

If the axis (ξ, η) are chosen such that $\psi_0 = 0$, the orbit takes the explicit form

$$r = \frac{p}{1 + e\cos\psi} \tag{1.131a}$$

where

$$p = \frac{L^2}{mk} \tag{1.131b}$$

and

$$e = \sqrt{1 + \frac{2EL^2}{mk^2}} \tag{1.131c}$$

Equations (1.131) represent a conic section of eccentricity e and *latus rectum* p with one focus on the origin. In our case $E < 0$ the conic is an ellipse with semi-axes

$$a = \frac{p}{1 - e^2} = \frac{k}{2E} \tag{1.132a}$$

$$b = \frac{p}{\sqrt{1 - e^2}} = \frac{L}{2m\,|E|} \tag{1.132b}$$

Figure 1.6 shows the elements of the keplerian ellipse in the plane.

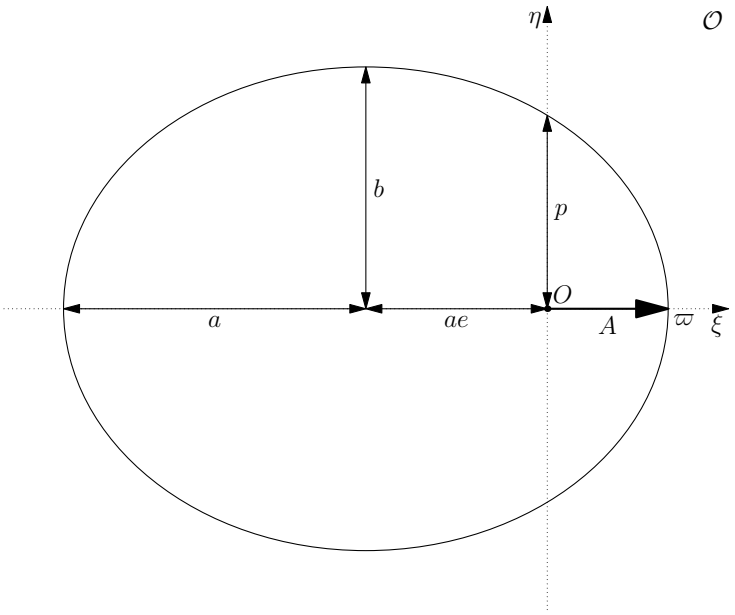

Figure 1.6: Elements of a keplerian ellipse

The integration of the time evolution equation (1.128) can be simplified very much if the physical constants E, L are expressed in term of the geometrical parameters a, e using equations (1.131c) and (1.132a). One obtains, then

$$t - t_0 = \sqrt{\frac{m}{k}} a^{\frac{1}{2}} \int \frac{r dr}{\sqrt{(ae)^2 - (r-a)^2}} \tag{1.133}$$

where the new integration constant

$$t_0 = \beta_E \tag{1.134}$$

is called the *perihelion passage time*. With the variable change

$$r = a(1 - e\cos u) \tag{1.135}$$

where u is the *eccentric anomaly*, the integral is elementary

$$t - t_0 = \frac{T}{2\pi}(u - e\sin u) \tag{1.136}$$

where we have introduced the *revolution period*

$$T = 2\pi\sqrt{\frac{m}{k}} a^{\frac{3}{2}} \tag{1.137}$$

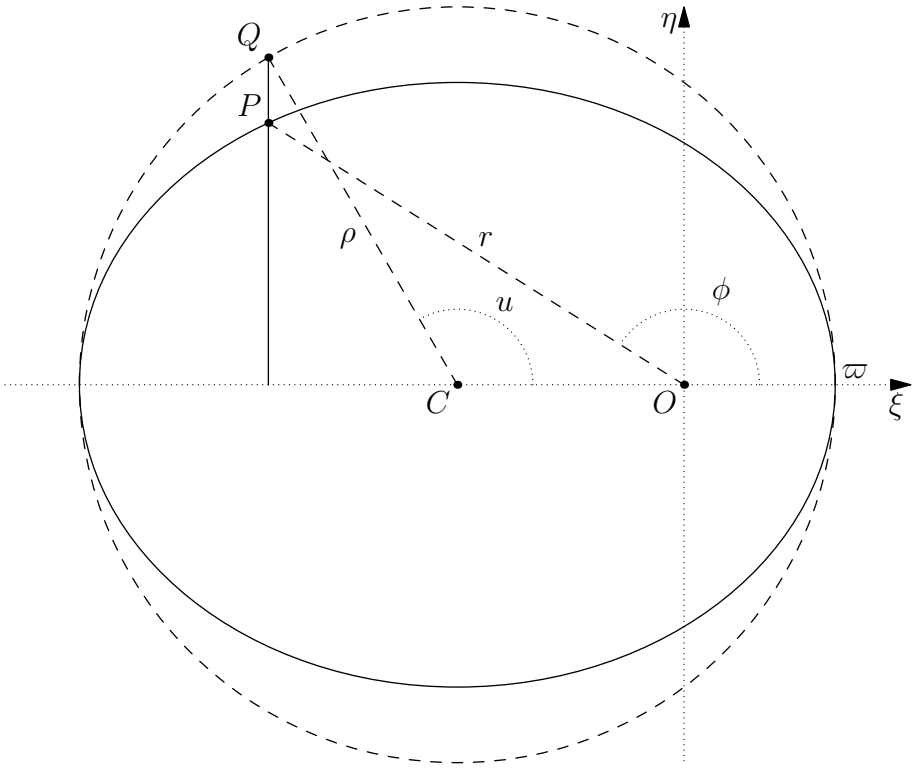

Figure 1.7: Angles u and ϕ in Keplerian motion.

Equation (1.136) is usually called the *Kepler equation* and together with (1.135) provide a parametric representation of the movement of a planet on its orbit.

Angle u has an interesting geometrical interpretation. Draw a circle of radius a, concentric with the orbit and mark a point Q on it, with the abscissa of the particle P. Let ρ be the distance between Q and C, then u is the angle between the auxiliary radius ρ and the direction of the pericenter ϖ.

1.6 Action-Angle variables

The full power of the Hamiltonian formulation is better seen when canonical variables suitable for specific applications are used. The canonical pair introduced above (Sect. 1.5) (α_i, β_i) are in general useful but they are not the

only possible set. For instance, any change of variables in the form

$$\gamma_i = \gamma_i(\alpha) = P'_i \tag{1.138}$$

yields another set of canonical variables where the Hamiltonian is only a function of the new momenta and the corresponding equations of motion can be simply integrated. Indeed

$$\dot{P'}_i = \frac{\partial H}{\partial \beta_i} = 0 \qquad P'_i = \gamma_i \qquad \dot{Q'}_i = \frac{\partial H}{\partial \gamma_i} = \omega_i \qquad Q'_i = \omega_i t + \beta_i \tag{1.139a}$$

For the particular case of periodic motion such a suitable set of canonical variables is the *action-angle variables*.

1.6.1 One-dimensional case

Let us point out that there are two types of periodic motion in a one dimensional Hamiltonian system: *rotation* and *libration*. In a rotation periodic motion, variable q can grow without limit as a function of time, although H is a periodic function of q, and the angular coordinate ϕ in a central potential is an example. The motion is a libration if coordinate q remains bounded. Both types of motion may be present in a mechanical system, depending on the initial condition, as in the case of a pendulum, with Hamiltonian[2]

$$H = \frac{p^2}{2l} + g(1 - \cos q) \tag{1.140}$$

which performs librations if $E < gl$ and rotations if $E > gl$ (Figure 1.8).

In a one dimensional Hamiltonian system, let us define the *action variable* J in the form

$$J = \frac{1}{2\pi} \oint p dq \tag{1.141}$$

where the integral runs though a full period of the system. Obviously, J does not depend on q and as the Hamiltonian is conserved

$$H(p, q) = \alpha_1 = E \qquad\qquad p = p(q.E)$$

J must be a function of E only

$$J = J(E) = J(\alpha_1) \qquad\qquad \alpha_1 = E = H(J) \tag{1.142}$$

[2]In next chapter the pendulum is extensively discussed.

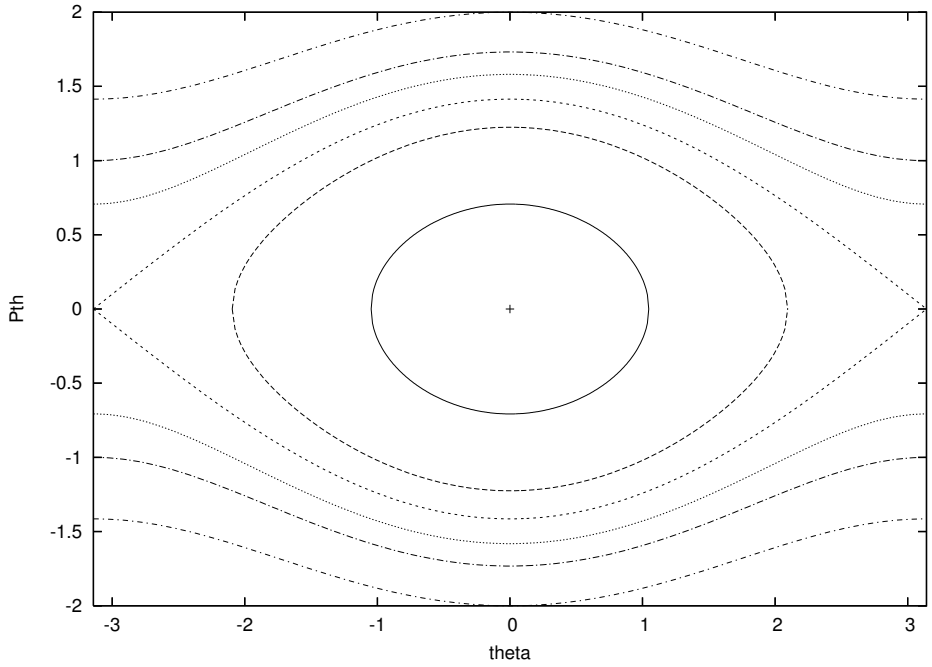

Figure 1.8: Phase space of a pendulum, with rotation and libration regimes (see next chapter).

Through this latter equation, the Hamilton characteristic function must be function of J

$$W = W(q, J) = F_2(q, J) \tag{1.143}$$

or, it can be interpreted as a F_2 type generating function of a canonical transformation to a new set of variables

$$P = J \qquad\qquad Q = \frac{\partial W}{\partial J} = w \tag{1.144}$$

and the new variable w is called the *angle variable*. This canonical transformation is of the type defined in (1.139) and w must be a linear function of time

$$w = \omega t + \beta \qquad\qquad \dot{w} = \frac{\partial H}{\partial J} = \omega \tag{1.145}$$

Among the (α, β) family of variables, action-angle variables enjoy certain advantages. First, the integration constant ω is the frequency of the periodic

motion, as can be easily seen computing the change in the angle variable along a period T of the motion

$$\Delta w = \frac{\partial \Delta W}{\partial J} = \frac{\partial (\oint p dq)}{\partial J} = 2\pi = \omega T \tag{1.146}$$

and the latter equality proves our statement.

Since the motion is periodic, any uniform function of (q, p) can be expanded in a Fourier series in w of period 2π

$$F(q,p) = \sum_{n=-\infty}^{\infty} F_n(J) e^{inw} \tag{1.147}$$

whose coefficients are functions of J only, a second important property of action-angle variables. As a particular case, (q, p) are uniform functions of (q, p), except coordinates themselves in the case of rotation, and can be expanded in the form

$$p = \sum_n p_n(J) e^{inw} \tag{1.148a}$$

$$q = \begin{cases} \sum_n q_n(J) e^{inw} & \text{libration} \\ \frac{w}{2\pi}\Delta q + \sum_n q_n(J) e^{inw} & \text{rotation} \end{cases} \tag{1.148b}$$

where Δq is the periodicity interval of variable q.

Let us compute the action-angle variables for the harmonic oscillator. The action variable is

$$\begin{aligned} J &= \frac{1}{2\pi} \oint p dq \\ &= \frac{1}{2\pi} \oint \sqrt{2mE - m^2\omega^2 q^2} dq \\ &= \frac{E}{\omega} \end{aligned} \tag{1.149}$$

the Hamiltonian and the (constant) frequency are

$$H = \omega J \qquad\qquad \omega = \frac{\partial H}{\partial J},$$

Hamilton principal function is

$$W = \sqrt{2m} \int \sqrt{\omega J - \frac{1}{2} m\omega^2 q^2} dq \tag{1.150}$$

and w is found from W

$$w = \frac{\partial W}{\partial J} = \arccos \sqrt{\frac{m\omega}{2J}} x \tag{1.151}$$

and finally, the Fourier series for the coordinates are

$$q = \sqrt{2J} m\omega \cos w \qquad\qquad p = \sqrt{2m\omega J} \sin w, \tag{1.152}$$

both having a single term.

1.6.2 The separable case

The results of Sect 1.6.1 can be extended immediately to a completely separable system of n variables. In that case

$$W = \sum_{i=1}^{n} W_i(q_i, \alpha) \tag{1.153}$$

where each W_i depends only on the q_i coordinate. The variables

$$J_i = \frac{1}{2\pi} \oint \frac{\partial W}{\partial q_i} dq_i = \frac{1}{2\pi} \oint p_i dq_i = J_i(\alpha) \tag{1.154}$$

are called *action variables* and are functions of the set of constants α. Inverting the set of equations (1.154) the generating function of a canonical transformation

$$W = \sum_{i=1}^{n} W_i(q_i, J) \tag{1.155}$$

and the new Hamiltonian

$$H = \alpha_1(J) = H(J) \tag{1.156}$$

are found. The *angle variables* are the canonical conjugates of the action variables

$$w_i = \frac{\partial W}{\partial J_i} \tag{1.157}$$

and the Hamilton equations take the simple form

$$\dot{J}_i = 0 \qquad\qquad \dot{w} = \frac{\partial H}{\partial J_i} = \omega_i \tag{1.158}$$

and it can be shown that the ω_i are the frequencies of the mechanical system.

Any uniform function of the canonical variables (q, p) can be expanded in a multiple Fourier series

$$F(q,p) = \sum_{\{k_1 \dots k_n\}} F_{\{k_1 \dots k_n\}}(J) e^{i \sum_{i=1}^{n} k_i w_i} \tag{1.159}$$

In the general case, a mechanical system with n degrees of freedom has n *independent frequencies*, that is to say, it does not exist a set of integers $N = \{k_1 \dots k_n\}$ such that the linear combination $\Omega = \sum_{i=1}^{n} k_i \omega_i(J) = 0$. Such a type of motion is called *multiply periodic* or also *quasiperiodic*. In the particular case where a set N exist such that $\Omega = 0$ the motion is *resonant*. The motion will be periodic only if all the frequencies ω_i are multiples of a single frequency ω_0 and in that case the system is resonant. If m frequencies are equal, the system is called $m-1$ *times degenerate* and if all frequencies are equal the system is *fully degenerate*[3].

In the case of resonance the problem can be simplified through the elimination of some of the angle variables. Indeed, let us introduce the generating function

$$F_2 = J_1' \sum_{i=1}^{n} k_i w_i + \sum_{j=2}^{n} J_j' w_j \tag{1.160}$$

and the new canonical variables are

$$w_j' = \begin{cases} \sum_{i=1}^{n} k_i w_i & j = 1 \\ w_j & j \geq 2 \end{cases} \tag{1.161a}$$

and

$$J_j' = \begin{cases} \frac{J_1}{n_1} & j = 1 \\ J_j - \frac{n_j}{n_1} J_1 & j \geq 2 \end{cases} \tag{1.161b}$$

and the w_1 variable is a (local in general) constant of motion.

[3]Resonances will be extensively discussed in the forthcoming chapters.

1.6.3 The Kepler problem

Let us find the action-angle variables for the Kepler problem. From the definition (1.154)

$$J'_3 = \frac{1}{2\pi}\oint p_\phi d\phi = L_z \tag{1.162a}$$

$$J'_2 = \frac{1}{2\pi}\oint p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}d\theta \tag{1.162b}$$

$$J'_1 = \frac{1}{2\pi}\oint p_r dr = \frac{1}{2\pi}\oint \sqrt{2mE + \frac{2mk}{r} - \frac{L^2}{r^2}}dr \tag{1.162c}$$

The first integral is immediate and the others can be easily computed with suitable changes of variables (Sect. 1.5.2) or with the powerful tools of complex integration. The results are

$$J'_3 = L_z \tag{1.163a}$$

$$J'_2 = L - L_z \tag{1.163b}$$

$$J'_1 = \frac{k}{2}\sqrt{\frac{2m}{-E}} - L \tag{1.163c}$$

and from these equations the set of constants of the motion $\alpha = (E, L, L_z)$ can be replaced by J_i set. The Hamiltonian is

$$H = -\frac{\frac{1}{2}mk^2}{(J'_1 + J'_2 + J'_3)^2} \tag{1.164}$$

From this expression we find that the system is completely degenerate with a frequency

$$\omega_1 = \omega_2 = \omega_3 = n = \frac{mk^2}{(J'_1 + J'_2 + J'_3)^3} \tag{1.165}$$

and the motion is periodic.

An easy way to compute the w_i variables is to observe that since the canonical conjugate variables to the α set is $\beta = (-\Omega, -\omega, -t_0)$ a canonical transformation leading from the α, β variables to the (J, w) ones is

$$F_2 = J'_3\Omega + (J'_2 + J'_3)\omega - \frac{\frac{1}{2}mk^2}{(J'_1 + J'_2 + J'_3)^2}(t - t_0) \tag{1.166}$$

from which we obtain the w variables in the form

$$w_3' = \Omega + n(t - t_0) \tag{1.167a}$$
$$w_2' = \omega + n(t - t_0) \tag{1.167b}$$
$$w_1' = n(t - t_0) \tag{1.167c}$$

and the latter expression is the *mean anomaly* of the orbit.

Since the system is fully degenerate, there are several canonical transformations that make two of the angle variables constants of the motion. One of them is generated by

$$F_2' = J_1 w_1' + J_2(w_2' - w_1') + J_3(w_3' - w_2') \tag{1.168}$$

which transforms the Kepler problem to the *Delaunay variables*

$$J_1 = \sqrt{mka} \qquad w_1 = n(t - t_0) \tag{1.169a}$$
$$J_2 = L \qquad w_2 = \omega \tag{1.169b}$$
$$J_3 = L_z \qquad w_3 = \Omega \tag{1.169c}$$

As we have shown before, any uniform function of the canonical variables can be expanded as a Fourier series in the Delaunay variables (or any variation of them). These expansions are the basis of perturbation theory in celestial mechanics.

Chapter 2

Non–linear Oscillations

The time evolution of any variable in a conservative Hamiltonian system exhibits, in a broad sense, oscillations. There is however a main difference between linear and non–linear oscillations, the latter appear in non-linear dynamical systems. A simple way to distinguish a linear system from a non-linear one is just by the inspection of the corresponding Hamiltonian differential equations. If they are linear in the coordinates, the system is called linear while if they involve non-linear functions of the coordinates, the system is said to be non-linear. The main difference between linear and non-linear systems is in the oscillation frequency. In general, a linear system presents a constant value of the frequency (the so-called isochronism), while in a non-linear one the frequency depends on the integrals of motion, like the energy for instance. This fact has a major implication when we model and study resonances in non–linear systems.

In this Chapter we introduce two examples of non-linear models of one degree of freedom which will be largely used along this text, just by considering perturbations to these models or constructing many dimensional systems by recourse of one of the systems presented in the last section of the current chapter.

2.1 The Pendulum

One of the simpler and at the same time fairly useful non–linear systems is the pendulum. Let us consider in the xy plane, a point mass m suspended from a pivot as shown in Figure 2.1. Therefore

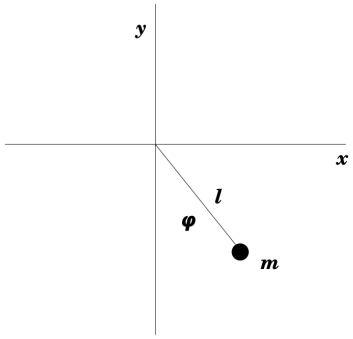

Figure 2.1: Sketch of the ideal pendulum

$$x = l\sin\varphi; \quad y = -l\cos\varphi,$$

where l is the constant length of the pendulum and φ the displacement of the point mass with respect to the y-axis. The potential energy is then given by

$$V = mgy = -mgl\cos\varphi.$$

Notice that the momentum conjugate to φ is $p = ml^2\dot{\varphi}$, so denoting $M = ml^2$ and $V_0 = mgl$ the corresponding 1D Hamiltonian can be recast as:

$$H(p,\varphi) = \frac{p^2}{2M} - V_0\cos\varphi. \tag{2.1}$$

The potential energy of the pendulum is displayed in Figure 2.2. For different energy values $H(p,\varphi) = h$, we observe that the minimum at $\varphi = 0$, $h = -V_0$, corresponds to the rest position at bottom, while for those values of φ for which $-V_0 < h < V_0$ there are turning points, so that the pendulum oscillates with an amplitude that increases with h, and for $h > V_0$, φ is not bounded and the pendulum rotates.

In most of the elementary books of mechanics, only the case of small oscillations is considered. That is, if $|\varphi| \ll 1$ which corresponds to $|h+V_0| \ll 1$, then

$$\cos\varphi = 1 - \frac{\varphi^2}{2} + \mathcal{O}(\varphi^4),$$

so, neglecting constants (or shifting $h \to h + V_0$) and retaining up to second order in φ, the Hamiltonian (2.1) reduces to

$$H_2(p,\varphi) = \frac{p^2}{2M} + V_0\frac{\varphi^2}{2},$$

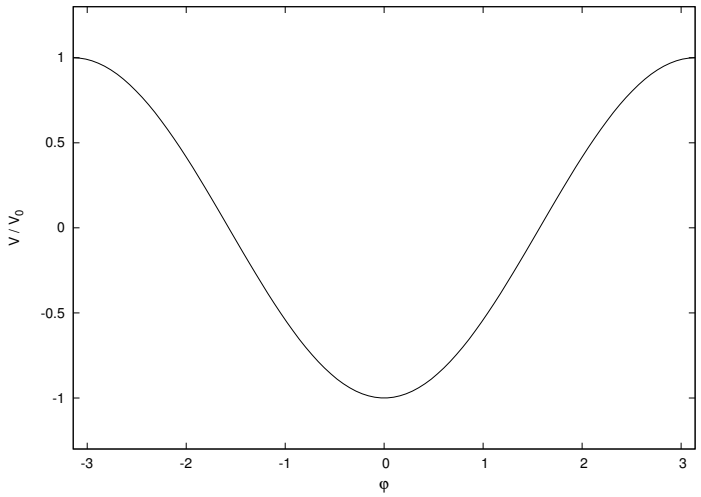

Figure 2.2: Potential energy of the pendulum

which corresponds to a harmonic oscillator, whose solution setting $\varphi(0) = 0$, is

$$\varphi(t) = \varphi_0(h) \sin \omega_0 t,$$

where $\varphi_0(h)$ is the (small) amplitude and $\omega_0^2 = V_0/M$ is the small oscillation frequency, *that it is constant*.

But we are interested in solving the pendulum for any value of the energy, so let us rescale the Hamiltonian (2.1) such that $M = 1$, so it could be written as

$$H(p, \varphi) = \frac{p^2}{2} - V_0 \cos \varphi, \qquad V_0 = \omega_0^2. \tag{2.2}$$

From the Hamilton equations $p = \dot{\varphi}$, then (2.2) for a given energy label h leads to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \pm\sqrt{2(h + V_0 \cos \varphi)}. \tag{2.3}$$

Thus, setting $\varphi = 0$ for $t = 0$, we can write

$$\pm t = \int_0^{\varphi} \frac{d\theta}{\sqrt{2(h + V_0 \cos \theta)}}. \tag{2.4}$$

Clearly this integral should have different solutions depending if $0 < h < V_0$ or $h > V_0$, since in the first case the argument inside the square root could be negative. Physically, it is simple to understand that in one case we would

have oscillations and in the other one rotations. The $\pm$ sign is only relevant for defining the sense of rotation of the pendulum. In any case, the solution of the integral (2.4) is in terms of elliptical functions.

2.1.1 Oscillations

Let us first focus in the case in which $h < V_0$, that is $|\varphi(t)| < \pi$. In any table of integrals it can be found that

$$t = \frac{1}{\sqrt{V_0}}\,\mathrm{F}(\gamma, r^{-1}), \tag{2.5}$$

where

$$\sin\gamma = \sqrt{\frac{V_0(1-\cos\varphi)}{h+V_0}},\;\; r = \sqrt{\frac{2V_0}{h+V_0}},\quad \mathrm{F}(\beta,k) = \int_0^\beta \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}. \tag{2.6}$$

The last expression in equation (2.6) is the elliptic function of the first kind.

In order to simplify the computation, let us consider the amplitude of oscillation $\varphi_0(h)$ instead of h. Clearly, from (2.1) the amplitude satisfies

$$h = -V_0\cos\varphi_0,$$

and, using the trigonometric relation

$$\sin^2\frac{\psi}{2} = \frac{1-\cos\psi}{2},$$

it is straightforward to show that

$$h + V_0 = 2V_0\sin^2\frac{\varphi_0}{2},$$

then

$$r^{-1} = \sin\frac{\varphi_0}{2} \equiv k(\varphi_0), \qquad \sin\gamma = \frac{\sin(\varphi/2)}{\sin(\varphi_0/2)}.$$

Recalling that $V_0 = \omega_0^2$ according to (2.2), then from equations (2.5) and (2.6) we obtain

$$\omega_0 t = \int_0^{\gamma(\varphi,\varphi_0)} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}. \tag{2.7}$$

The solution for $\gamma(\varphi, \varphi_0)$ is given in terms of the Jacobi elliptical amplitude:

$$\gamma(\varphi, \varphi_0) = \mathrm{am}(\omega_0 t, k(\varphi_0)), \quad \rightarrow \quad \sin\gamma = \sin(\mathrm{am}(\omega_0 t, k)) = \mathrm{sn}(\omega_0 t, k),$$

where $\mathrm{sn}(u)$ is the so–called Jacobi elliptical sine. Thus, for φ we attain the following implicit solution:

$$\sin\frac{\varphi}{2} = \sin\frac{\varphi_0}{2}\,\mathrm{sn}(\omega_0 t, k) \equiv k\,\mathrm{sn}(\omega_0 t, k).$$

In order to get an explicit solution for $\varphi(t)$, let us take the time derivative of the last expression:

$$\frac{\dot\varphi}{2}\cos\frac{\varphi}{2} = k\omega_0 \frac{d}{d(\omega_0 t)}\mathrm{sn}(\omega_0 t, k),$$

using the relationship

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn}(u,k) = \mathrm{cn}(u,k)\mathrm{dn}(u,k), \quad \mathrm{dn}(u,k) = \sqrt{1-k^2\mathrm{sn}^2(u)},$$

where $\mathrm{cn}(u,k)$ stands for the Jacobi elliptic cosine, we get

$$\frac{\dot\varphi}{2}\cos\frac{\varphi}{2} = k\omega_0\,\mathrm{cn}(\omega_0 t, k)\mathrm{dn}(\omega_0 t, k).$$

But since

$$k^2\mathrm{sn}^2(\omega_0 t, k) = \sin^2\frac{\varphi}{2} \qquad \text{then} \qquad \mathrm{dn}(\omega_0 t, k) = \cos\frac{\varphi}{2},$$

we arrive to an explicit solution for $\dot\varphi$,

$$\dot\varphi(t) = 2k\omega_0\mathrm{cn}(\omega_0 t, k).$$

The Fourier series expansion for $\mathrm{cn}(u,k)$[1] is given by

$$\mathrm{cn}(u,k) = \frac{2\pi}{k\mathrm{K}(k)}\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1+q^{2n-1}}\cos\left((2n-1)\frac{\pi u}{2\mathrm{K}(k)}\right);$$

with

$$\mathrm{K}(k) = \mathrm{F}\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

[1]See Gradshteyn and Ryzhik, Table of Integrals, Series, and Products, Seventh Edition (2007), for the strip of convergence of this expansion.

the complete elliptical integral of first kind, and

$$q = e^{\pi \frac{\mathrm{K}'}{\mathrm{K}}}, \quad \mathrm{K}' = \mathrm{F}\left(\frac{\pi}{2}, k'\right), \quad k' = \sqrt{1-k^2}.$$

Thus the coefficients of the Fourier series can be written as

$$\frac{q^{n-1/2}}{1+q^{2n-1}} = \frac{1}{q^{n-1/2}+q^{-(n-1/2)}} = \frac{1/2}{\cosh\left(\mathrm{K}'(2n-1)\frac{\omega(k)}{\omega_0}\right)},$$

where

$$\omega(k) = \frac{\pi\omega_0}{2\mathrm{K}(k)}, \tag{2.8}$$

so, finally there results

$$\dot{\varphi}(t) = 4\omega(k)\sum_{n=1}^{\infty}\frac{1}{\cosh\left(\mathrm{K}'(2n-1)\frac{\omega(k)}{\omega_0}\right)}\cos\left((2n-1)\,\omega(k)\,t\right). \tag{2.9}$$

We should remark that among all the terms involved, the oscillations have a frequency $\omega(k)$ $\left(k = \sin\varphi_0/2 = \sqrt{(h+V_0)/2V_0}\right)$, given by (2.8), that depends on the energy. This is in fact the non–linear character of the pendulum oscillations.

Since $\dot{\varphi}(t)$ is an analytic function, we can integrate (2.9) term by term to obtain $\varphi(t)$. Denoting by $\omega_n(k) = (2n-1)\omega(k)$, a simple calculation leads to

$$\varphi(t) = 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cosh\left(\mathrm{K}'(k)\frac{\omega_n(k)}{\omega_0}\right)}\sin\left(\omega_n(k)\,t\right). \tag{2.10}$$

Thus we have obtained an explicit form for $\varphi(t)$ in terms of a Fourier series, where both, the coefficients and the frequency depends on the energy h through the parameter k. Notice that the frequency spectrum involves solely odd terms.

In order to check the result given in (2.10) and to illustrate the manipulation of elliptic integrals, let us consider that $h \simeq -V_0$, that is, the small oscillation regime. Recalling the expression for k, for these small values of the energy, $k \ll 1$, and also $\varphi_0 \ll 1$. We start with the frequency (2.8), so we need an approximate expression for $\mathrm{K}(k)$. On expanding the argument of the elliptic integral of the first kind in powers of k^2 up to $\mathcal{O}(k^2)$, we obtain

$$\frac{1}{\sqrt{1-k^2\sin\alpha}} = 1 + \frac{1}{2}k^2\sin^2\alpha + \mathcal{O}(k^4),$$

and then, up to order k^2, $K(k)$ and $\omega(k)$ reduce to

$$\mathrm{K(k)} = \frac{\pi}{2}\left(1 + \frac{\mathrm{k}^2}{4}\right), \qquad \omega(\mathrm{k}) = \omega_0\left(1 - \frac{\mathrm{k}^2}{4}\right).$$

From the above expression, we observe that the first correction to the small oscillation frequency decreases linearly with h. Now, for the coefficients in (2.10),

$$a_n(k) = \frac{4}{(2n-1)\cosh\left(\mathrm{K}'(k)\frac{\omega_n(k)}{\omega_0}\right)},$$

we use[2]

$$\mathrm{K}'(k) = \ln\frac{4}{k} + \mathcal{O}(k^4),$$

and the hyperbolic cosine can be approximated by

$$\cosh\left(\mathrm{K}'(k)\frac{\omega_n(k)}{\omega_0}\right) \approx \frac{1}{2}\left(\left(\frac{4}{k}\right)^{(2n-1)(1-k^2/4)} + \left(\frac{k}{4}\right)^{(2n-1)(1-k^2/4)}\right).$$

Then, since $k \ll 1$, the second term in the sum can be neglected and keeping only up to $\mathcal{O}(k)$, that corresponds to the small oscillation regime, due to the fact that the amplitude depends on $\sqrt{h}$, only one term in the series should be retained – the first one with $n = 1$–, and the corresponding coefficient is

$$a_1 \approx 2k = 2\sin\frac{\varphi_0}{2} \approx \varphi_0, \qquad k,\, \varphi_0 \ll 1,$$

and for the same order in k, $\omega(k) \approx \omega_0$, thus

$$\varphi(t) \approx \varphi_0 \sin\omega_0 t,$$

which in fact coincides with the solution for the harmonic oscillator.

2.1.2 Rotations

Now let us consider the case $h > V_0$, in which $\varphi(t) \in (0, 2\pi)$. The solution for (2.4) corresponds to

$$t = \frac{2}{\sqrt{2(h+V_0)}}\,\mathrm{F}(\varphi/2, r^{-1}), \tag{2.11}$$

[2]See Gradshteyn and Ryzhik (2007).

where

$$r = \sqrt{\frac{h + V_0}{2V_0}}$$

and $\mathrm{F}(\beta, k)$ is the same function introduced in (2.6). Defining

$$k^2 = \frac{2V_0}{h + V_0} \left(= \frac{1}{k_{\mathrm{osc}}^2} \right), \qquad \omega_r^2(h) = \frac{h + V_0}{2} = \frac{V_0}{k^2} = \frac{\omega_0^2}{k^2},$$

we see again that the dependence on h is introduced through the parameter k. So, we can write

$$\omega_r(k)t = \int_0^{\varphi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}},$$

then, using the definition of the Jacobian elliptical amplitude, we get an explicit solution for $\varphi(t)$:

$$\pm\varphi(t) = 2\mathrm{am}(\omega_r(k)t, k).$$

Now, we use the Fourier expansion of $\mathrm{am}(u, k)$[3]:

$$\mathrm{am}(u, k) = \frac{\pi u}{2\mathrm{K}(k)} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 + q^{2n}} \sin\left(\frac{n\pi u}{\mathrm{K}(k)} \right),$$

where q denotes the same quantity defined before for the expansion of $\mathrm{cn}(u, k)$. Introducing the half–frequency of rotation

$$\omega(k) = \frac{\pi \omega_r(k)}{2\mathrm{K}(k)}, \tag{2.12}$$

and denoting with $\omega_n(k) = 2n\omega(k)$, the expansion for $\varphi(t)$ in the case of rotations is

$$\pm\varphi(t) = 2\omega(k)t + 4 \sum_{n=1}^{\infty} \frac{1}{2n \cosh\left(n\pi \frac{\mathrm{K}'(k)}{\mathrm{K}(k)} \right)} \sin\left(\omega_n(k)\, t \right). \tag{2.13}$$

Again, we see that the frequency (or half–frequency) depends strongly on the energy and, in this case, only do the even harmonics appear.

[3] See again the strip of convergence of this expansion in Gradshteyn and Ryzhik (2007)

Let us consider the limiting case for which $h \gg V_0$, that would correspond to a slightly perturbed free rotator. In this case

$$k \ll 1, \qquad \mathrm{K(k)} \approx \frac{\pi}{2}\left(1+\frac{\mathrm{k}^2}{4}\right) \qquad \omega(\mathrm{k}) = \omega_{\mathrm{r}}\left(1-\frac{\mathrm{k}^2}{4}\right),$$

so at order k, $\omega(k) = \omega_r(k)$ and using the same approximation for $\mathrm{K}'(k)$ in the case of $k \ll 1$,

$$\cosh\left(n\pi\frac{\mathrm{K}'(k)}{\mathrm{K}(k)}\right) \approx \frac{1}{2}\left(\frac{4}{k}\right)^{2n(1-k^2/4)},$$

then at order k in (2.13), the first term for $n = 1$ is of order $\mathcal{O}(k^2)$, so it reduces to

$$\varphi(t) \approx \pm 2\omega_r(k)t,$$

which is the well–known solution for a free rotator, where the $p = \sqrt{2h} \approx 2\omega_r$.

2.1.3 The separatrix

We have already considered both cases $h < V_0$ (oscillations) and $h > V_0$ (rotations); now we will focus in the case $h = V_0$. In such a case Eq. (2.2), $H(p, \varphi) = V_0$ seems to have "several" possible solutions. Indeed,

$$H(p,\varphi) = \frac{p^2}{2} - V_0\cos\varphi = V_0 \equiv \omega_0^2$$

has as solutions, $p = 0, \varphi = \pm\pi$, that correspond to the pendulum at the top, and using the trigonometric relation between φ and $\varphi/2$, the "other" solution is

$$p_s = \pm 2\omega_0 \cos\left(\frac{\varphi_s}{2}\right), \tag{2.14}$$

where the subscript s refers to the separatrix solution. Clearly, from Fig.2.2, the energy level $h = V_0$ separates oscillations from rotations, and for this reason this curve in the (p, φ) plane is called *separatrix*. Taking into account that $p_s = \dot{\varphi}_s$, allowing $-\infty < t < \infty$, we can integrate (2.14) and get

$$\varphi_s(t) = 4\arctan\left(e^{\omega_0 t}\right) - \pi. \tag{2.15}$$

Clearly, this trajectory is not periodic or instead, it has an infinite period since

$$\lim_{t\to\infty}\varphi_s(t) = \pi, \qquad \lim_{t\to-\infty}\varphi_s(t) = -\pi.$$

This means that the separatrix, given by (2.15), is an asymptotic trajectory that approaches $\pm\pi$ as $t \to \pm\infty$. In other words, the separatrix connects the points $p = 0, \varphi = \pm\pi$ in the (p, φ) plane. In order to learn more details about the real nature of the separatrix and its relationship with the points $p = 0, \varphi = \pm\pi$, let us bring our attention to the so–called fixed points of the Hamiltonian.

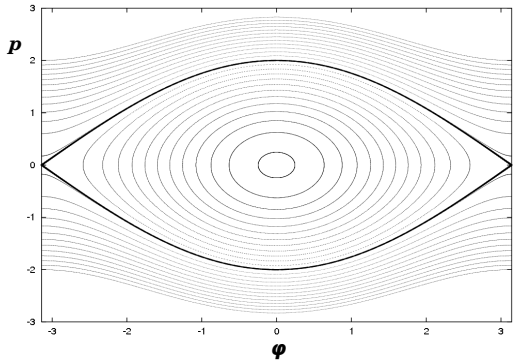

Figure 2.3: Phase space structure of the pendulum setting $V_0 = 1$. Each curve is parameterized by the energy h. The dark line corresponds to the separatrix ($h = 1$).

2.1.4 Fixed Points and their Stability

By definition, a fixed point satisfies

$$\dot{p} = -\frac{\partial H}{\partial \varphi} = 0, \qquad \dot{\varphi} = \frac{\partial H}{\partial p} = 0. \tag{2.16}$$

That is the Hamiltonian flow is null, which means that for any fixed point the system is at rest. Physically, from Fig. 2.2 the fixed points are those where the line of constant energy h intersects the potential function $V(\varphi)$ at a single point. From Fig. 2.3 it is evident that the phase space of the pendulum is a cylinder. The sides $\varphi = -\pi$ and $\varphi = \pi$ should be identified, so both points are the same. Thus, it becomes clear that the fixed points are $p = 0, \varphi = 0$, for which the pendulum is at rest at the bottom, and $p = 0, \varphi = \pm\pi$, which correspond to the pendulum at rest on top.

Anyway it would be instructive to derive the fixed points and their linear stability from (2.16). To that end let us denote the phase φ by x, and

$$\dot{p} = v(x,p) \qquad \dot{x} = u(x,p). \tag{2.17}$$

If (x_0, p_0) is a fixed point, it is $v(x_0,p_0) = 0$, $u(x_0,p_0) = 0$. On introducing

$$\xi = x - x_0, \qquad \eta = p - p_0, \qquad |\xi|, |\eta| \ll 1,$$

from (2.17), we have

$$\dot{\xi} = u(x_0 + \xi, p_0 + \eta), \qquad \dot{\eta} = v(x_0 + \xi, p_0 + \eta).$$

Expanding these expressions around (x_0, p_0) up to first order in ξ, η we get

$$\dot{\xi} = u_x^0 \xi + u_p^0 \eta, \qquad \dot{\eta} = v_x^0 \xi + v_p^0 \eta, \tag{2.18}$$

where the subscripts x, p denote the derivative respect to those variables and the superscript 0, that such derivatives are evaluated at the fixed point.

The last set of equations can be rewritten in the form

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} u_x^0 & u_p^0 \\ v_x^0 & v_p^0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \text{or} \qquad \dot{\boldsymbol{\delta}} = \Lambda^0 \boldsymbol{\delta}, \tag{2.19}$$

where $\boldsymbol{\delta}$ is the small deviation vector from the fixed point and Λ^0 is the 2×2 real matrix defended by the derivatives of the Hamiltonian flow evaluated at the fixed point. Clearly, the linear stability of the fixed point is completely determined by the eigenvalues of Λ^0. Real eigenvalues leads to unstable fixed points and imaginary ones to stable fixed points.

For the pendulum, as we have already mentioned, we have two fixed points: $p = 0, \varphi = 0, \pi$. Let us focus on the point $(0, \pi)$, which we will show is linearly unstable and we let for the reader the analysis or the other one (that in fact is linearly stable). For the pendulum Hamiltonian (2.2)

$$u_x^0 = 0, \quad u_p^0 = 1, \quad v_x^0 = V_0 = \omega_0^2, \quad v_p^0 = 0.$$

Therefore, the corresponding eigenvalues for Λ^0 are

$$\det\left(\Lambda^0 - \lambda I\right) = 0, \qquad \lambda = \pm\omega_0.$$

Let us denote with e_+ and e_- the associated eigenvectors corresponding to $\lambda = \omega_0$ and $\lambda = -\omega_0$ respectively, whose components in the basis $\mathcal{B} = \{n_1, n_2\}$ are

$$e_+ = \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix}, \qquad e_- = \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}. \tag{2.20}$$

Then from the eigenvectors equation we get

$$\Lambda^0 e_\pm = \pm\omega_0 e_\pm$$

and then for the components $\xi_\pm, \eta_\pm$ we get the relation (eigenslopes)

$$\frac{\eta_\pm}{\xi_\pm} = \pm\omega_0.$$

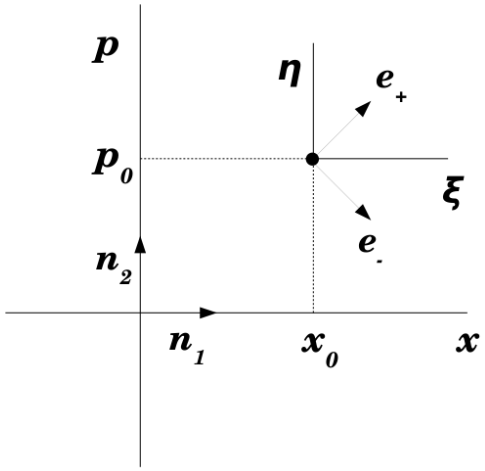

Figure 2.4: Unstable fixed point and the corresponding eigenvectors.

This is shown schematically in Fig. 2.4, for a generic unstable fixed point. Now if we introduce a change of basis $\mathcal{B} \to \mathcal{B}' = \{e_+, e_-\}$, the matrix Λ^0 takes the form

$$\Lambda^0 = \begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix}. \tag{2.21}$$

Thus denoting with $\bar{\xi}$ and $\bar{\eta}$ the components of the vector $\boldsymbol{\delta}$ in the basis $\mathcal{B}'$, (2.19) has as simple solutions

$$\bar{\xi}(t) = \bar{\xi}_0 e^{\omega_0 t}, \qquad \bar{\eta}(t) = \bar{\eta}_0 e^{-\omega_0 t}. \tag{2.22}$$

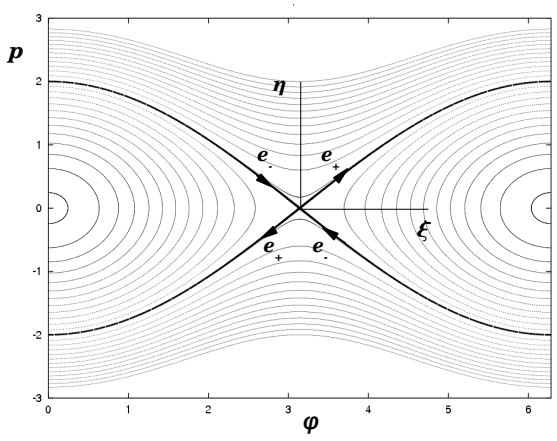

Figure 2.5: Stable and unstable manifold and the corresponding dynamics around the unstable fix point.

Considering initial conditions $\bar{\eta}_0 = 0$ and $|\bar{\xi}_0| \neq 0$ but small, we see from (2.22) that as $t \to \infty$, $\bar{\xi}(t) \to \infty$, that is along the direction e_+, any point moves away from the fixed point. On the other hand, for $|\bar{\eta}_0| \neq 0$ and $\bar{\xi}_0 = 0$, along the direction e_-, as $t \to \infty$, $\bar{\eta}(t) \to 0$, and every point moves towards the fixed point. Therefore as expected, the fixed point $p = 0, \varphi = \pi$ is unstable and as we see from Fig.2.5, the dynamics around this point is hyperbolic. The eigenvector e_+ is tangent to the so–called unstable manifold, and e_- is tangent to the corresponding stable manifold. As we can clearly observe those manifolds are the two different branches of the separatrix, the positive and negative one. As it is clear from the figure the upper branch of the separatrix only exists for positive values of p, while the lower one for negative p values. Thus at the unstable fixed point, they keep the direction of motion, as the arrows indicate. They do not cross, we can say that the unstable fixed point shows up when the stable and unstable manifolds intersect each other.

If we let $-\infty < t < \infty$ and taking into account that the phase space of the pendulum is a cylinder, in Fig.2.6 we present a sketch of how the unstable manifold W^u matches the stable manifold W^s at the point Q.

In the language of Hamiltonian systems, since the pendulum is an integrable system, every curve of the phase portrait shown in Fig.2.3 is said to be a 1D torus. Each of them is parameterized by the corresponding value of the energy h or the corresponding action or frequency. In the case of the

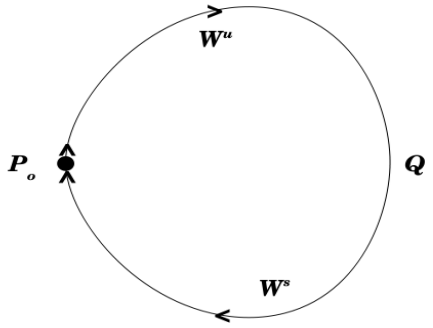

Figure 2.6: Sketch of how the unstable manifold W^u matches the stable manifold W^s at the point Q.

fixed points, it is customary to say that located at $p = 0, \varphi = 0$ there is a stable torus (or elliptic) of dimension 0, while that located at $p = 0, \varphi = \pi$, is also of dimension 0 (because both are just a point not a curve), and the latter is usually called *whiskered torus*, since it could be thought of as the intersection of two *whiskers* the stable and the unstable ones. In this particular case, both whiskers match exactly and we can define the separatrix. In fact, in the sketch in Fig.2.6, it is possible to draw a curve departing from and arriving at the unstable fixed point, because both manifolds coincide at the point Q. Thus the unstable fixed point, in this case, is called the *whiskered torus*, resembling the classical whiskers (see Fig.2.5) however, this concept could be better perceived in higher dimensions, as we shall see later.

One of the more important results regarding the pendulum is its frequency of motion. In Fig.2.7 we display how ω depends on the energy, just computing (2.8) and (2.12) after shifting $h \to h - V_0$ such that the energy of the separatrix corresponds to $h = 0$. Note that for rotations we have defined the half–frequency in order to avoid a jump ~ 2 in the transition from oscillations to rotations, as it can be seen in Fig.2.3 for the phase space portrait of the pendulum, close to the separatrix.

From this figure we see that for the oscillation regime ω is a decreasing function of h, having as upper bound the value ω_0. Close to $h = 0$, that in this case corresponds to the separatrix energy, the frequency decays rather fast to 0, which indeed is the expected behavior since the period of the separatrix tends to ∞. For the rotation regime, ω increases monotonically with h and

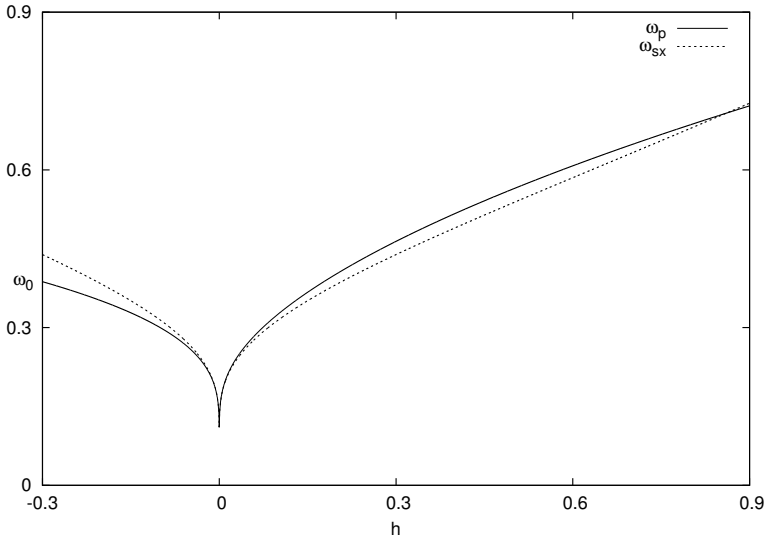

Figure 2.7: Dependence of the pendulum frequency (ω_p in the figure) with the energy, h, setting $V_0 = \omega_0^2 = 0.15$ and shifting the energy values such that the energy of the separatrix is 0. The dotted line represents the asymptotic value of the frequency in the vicinity of the separatrix, ω_{sx} (see next Section).

for large energies it behaves as $\sim \sqrt{h}$, since the pendulum approaches a free rotator, as already discussed.

2.1.5 Motion in the vicinity of the separatrix

The particular behavior of the frequency close to the separatrix suggests to study the motion at energies $h \approx V_0$. Let us rescale the energy to an adimensional one w in the fashion

$$w = \frac{h - V_0}{V_0},$$

which measures the relative distance to the separatrix, and consider the case $|w| \ll 1$. Clearly $w < 0$ corresponds to oscillations and $w > 0$ to rotations, while $w = 0$ is the value for the separatrix.

For oscillations we have already found that the frequency is

$$\omega(k) = \frac{\pi\omega_0}{2\mathrm{K}(k)},$$

where

$$k^2 = \frac{h + V_0}{2V_0}.$$

A simple manipulation allows us to write

$$k^2 = 1 + \frac{w}{2} = 1 - \frac{|w|}{2} \qquad \text{for} \qquad w < 0.$$

Now, using the approximation of $\mathrm{K}(k)$ for

$$k'^2 = 1 - k^2 = \frac{|w|}{2} \ll 1, \qquad \mathrm{K}(k) = \ln \frac{4}{k'} + \mathcal{O}(k'^4),$$

we can write it in terms of w

$$\mathrm{K}(w) \approx \frac{1}{2} \ln \frac{32}{|w|},$$

so the frequency reduces to

$$\omega(w) = \frac{\pi \omega_0}{\ln \frac{32}{|w|}}, \qquad \text{where} \qquad \lim_{w \to 0} \omega(w) = 0. \tag{2.23}$$

Now, for rotations ($w > 0$) the half–frequency of the motion is

$$\omega(k) = \frac{\pi \omega_r(k)}{2\mathrm{K}(k)},$$

where

$$\omega_r^2 = \frac{h + V_0}{2}, \qquad \text{and} \qquad k^2 = \frac{2V_0}{h + V_0} = \frac{\omega_0^2}{\omega_r^2}.$$

Thus,

$$k^2 = \frac{1}{1 + \frac{w}{2}} \approx 1 - \frac{w}{2}, \qquad w \ll 1,$$

so, since w is positive we can write

$$k^2 \approx 1 - \frac{|w|}{2}$$

and we obtain for k the same relationship with w as that for the case of oscillations. Since $\omega_0^2 = k^2 \omega_r^2$, for $k \approx 1$ $\omega_0 \approx \omega_r$. Therefore, we recover the same expression (2.23) for rotations in the vicinity of the separatrix.

On the other side, it is not difficult to show that the expression for $\varphi(t)$ is rather similar to those obtained previously. So we can conclude that the motion in a small neighborhood of the separatrix should not differ significantly from that on the separatrix itself, except that $\omega \to 0$ as the inverse of the logarithm of the energy. Fig.2.7 shows the behavior of ω given by (2.23), where we see the way in which the frequency goes to zero in the separatrix. Moreover this asymptotic value is also a good approximation for $|w| \sim 1$. It is clear from (2.23) that

$$\lim_{w\to 0^+} \frac{d\omega}{dw} = \lim_{w\to 0^-} \frac{d\omega}{dw} = \infty$$

so, the nonlinearity of the motion close to the separatrix, measured by $d\omega/dw$ is extremely large.

2.2 The pendulum frequency for low energies

We have already found that the oscillations of the pendulum has as fundamental frequency

$$\omega(k) = \frac{\pi\omega_0}{2K(k)}, \tag{2.24}$$

where ω_0 is the (constant) small oscillation frequency and $K(k)$ is the complete elliptical integral of first kind. Recalling that the amplitude of the oscillation, φ_0, is related with the energy h by

$$h = -V_0 \cos\varphi_0 = -V_0 \left(1 - 2\sin^2\frac{\varphi_0}{2}\right) = -V_0 + 2V_0 \sin^2\frac{\varphi_0}{2},$$

then

$$h + V_0 = 2V_0 \sin^2\frac{\varphi_0}{2} = 2\omega_0^2 k^2,$$

recalling that by setting $M = 1$, $V_0 = \omega_0^2$, and by definition $k^2 = \sin^2(\varphi_0/2)$.

If we shift the origin of the energy to $h + V_0 = h'$, and denoting again the energy with h instead of h' we get

$$h = 2\omega_0^2 k^2.$$

Now $K(k)$ is given by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{\sqrt{1 - k^2\sin^2\alpha}}. \tag{2.25}$$

Let

$$g(\alpha, k) = \frac{1}{\sqrt{1 - k^2 \sin^2 \alpha}},$$

and assume that $k^2 \ll 1$, small oscillations regime, so $h \ll 1$. We Taylor expand g up to first order in k^2

$$g(\alpha, k) = g(\alpha, 0) + g'(\alpha, 0)k^2 + \mathcal{O}(k^4),$$

where

$$g(\alpha, 0) = 1, \quad g'(\alpha, 0) \equiv \left(\frac{\mathrm{d}g}{\mathrm{d}k^2}\right)_{k^2=0} = \left(\frac{\sin^2 \alpha}{2(1 - k^2 \sin^2 \alpha)^{3/2}}\right)_{k^2=0} = \frac{\sin^2 \alpha}{2},$$

then

$$g(\alpha, k) \approx 1 + \frac{k^2}{2} \sin^2 \alpha$$

and therefore by (2.25)

$$K(k) \approx \frac{\pi}{2} + \frac{k^2}{2} \int_0^{\frac{\pi}{2}} \sin^2 \alpha \mathrm{d}\alpha = \frac{\pi}{2} + \frac{k^2}{2} \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{2} \left(1 + \frac{k^2}{4}\right).$$

Replacing this expression for $K(k)$ in (2.24) we obtain for the nearly linear frequency

$$\omega(k) \approx \frac{\pi \omega_0}{\pi \left(1 + \frac{k^2}{4}\right)} \approx \omega_0 \left(1 - \frac{k^2}{4}\right).$$

Using the relation $k^2 = h/2\omega_0^2$, we finally get for the frequency as a function of the energy in the vicinity of the stable equilibrium point

$$\omega(h) \approx \omega_0 - \frac{h}{8\omega_0}. \tag{2.26}$$

Therefore we see that within the oscillation regime $\omega(h) \leq \omega_0$. When considering higher orders term in k^2, it is easy to show that

$$\omega(h) \approx \omega_0 - \frac{h}{8\omega_0} - \frac{5}{256} \frac{h^2}{\omega_0^2} - \frac{11}{2048} \frac{h^3}{\omega_0^5}. \tag{2.27}$$

The fist term represent the linear oscillations while the rest of the terms correspond to the non–linear oscillations of the pendulum. The dependence

of ω with h leads to the nonlinearity of the motion and a way to measure it is, for instance, by

$$\frac{\mathrm{d}\omega}{\mathrm{d}h}.$$

In a 1D autonomous Hamiltonian system it is always possible to perform a canonical transformation to action–angle variables, so the Hamiltonian reads, $H(I) = h$, where I is the corresponding action. Clearly the above derivative should provide the very same information than

$$\frac{\mathrm{d}\omega}{\mathrm{d}I} = \frac{\partial^2 H}{\partial I^2}.$$

In this direction, Chirikov introduced the (adimensional) nonlinearity parameter as

$$\alpha = \frac{I}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}I}$$

that will help us latter when discussing, for instance, the non–linear resonance.

2.3 The Quartic Oscillator

In this Section we will consider another non–linear system that will serve as a model to several applications along this text. The system is the quartic oscillator whose Hamiltonian is

$$H(p, x) = \frac{p^2}{2} + \frac{x^4}{4}. \tag{2.28}$$

Adopting any energy label h, we can relate again the energy with the amplitude of oscillation a,

$$h = \frac{a^4}{4}.$$

Therefore, from (2.28) $\dot{x} = p$ and setting $x = 0$ at $t = 0$, we can write

$$t = \int_0^x \frac{dy}{\sqrt{(a^4 - y^4)/2}}. \tag{2.29}$$

The motion is possible within the potential $V(x) = x^4/4$, so only oscillations are possible and (2.29) has a single solution, which is in terms of the same

elliptic functions and integrals we have already found for the pendulum, in (2.5) and (2.6),

$$t = \frac{1}{a}\mathrm{F}\left(\delta, \frac{1}{\sqrt{2}}\right), \tag{2.30}$$

where

$$\cos\delta = \frac{x}{a}$$

Thus, since

$$at = \mathrm{F}\left(\delta, \frac{1}{\sqrt{2}}\right), \qquad \text{so} \qquad \delta = \mathrm{am}(at),$$

we obtain

$$\cos\delta = \cos(\mathrm{am}(at)) = \mathrm{cn}(at) \qquad \text{and} \qquad x(t) = a(h)\mathrm{cn}(at). \tag{2.31}$$

Taking the Fourier expansion of $\mathrm{cn}(at)$ and using the the energy again instead of the amplitude as a parameter, $a = \sqrt{2}h^{1/4}$, we finally get

$$x(t) = x_0(h)\sum_{n=1}^{\infty}\alpha_n \cos\left((2n-1)\sqrt{2}\beta h^{1/4}t\right), \tag{2.32}$$

where

$$\begin{aligned} x_0 &= 4\beta h^{1/4} = 2^{3/2}\beta a(h), \qquad \beta = \frac{\pi}{2\mathrm{K}(1/\sqrt{2})}, \\ \alpha_n &= \frac{1}{\cosh((n-1/2)\pi)}. \end{aligned} \tag{2.33}$$

Therefore, the oscillation frequency of the quartic oscillator is

$$\omega(h) = \sqrt{2}\beta h^{1/4} = \beta a(h). \tag{2.34}$$

The coefficients α_n satisfy the relationship

$$\frac{\alpha_{n+1}}{\alpha_n} \approx e^{-\pi} \approx \frac{1}{23}. \tag{2.35}$$

Thus, this model is clearly non–linear and has the interesting property that the Fourier coefficients decrease as fast as powers of $1/23^{(2(n-1))}$ for $n > 1$. On the other hand, the Hamiltonian flow has only one fixed point, at $p = 0, x = 0$

and it is evident that it is a stable one. The phase space portrait of the quartic oscillator is rather simple and it is left as an exercise to the reader.

Finally, from (2.32)-(2.35) we can write

$$x(t) \approx x_0(h)\,\alpha_1 \cos[\omega(h)t] = 2^{3/2}\,\beta\,a(h)\,\alpha_1\,\cos\vartheta, \qquad \vartheta \in \mathbb{S}^1,$$

where ϑ is the angle variable conjugated to the action I, $\alpha_1 \approx 0.4$ and thus $2^{3/2}\beta\alpha_1 \approx 0.96$. Moreover using the relation (Hamilton equation in action-angle variables for $\dot{\vartheta} = \omega$)

$$\omega(h) = \frac{dh}{dI},$$

it is rather simple to show that

$$I = \frac{a^3(h)}{3\beta},$$

and that the Hamiltonian in terms of action-angle variables $\mathcal{H} \equiv h$ takes the form

$$\mathcal{H}(I) = AI^{4/3}, \qquad A = \left(\frac{3\beta}{2\sqrt{2}}\right)^{4/3},$$

independent of the angle variable, as expected. Therefore, in terms of action-angle variables a simple approximate solution for the quartic oscillator takes the form

$$x(t) \approx a(I)\cos\vartheta, \qquad a(I) = CI^{1/3},$$

where $C = 2^{3/2}3^{1/3}\beta^{4/3}\alpha_1 \approx 1.3$, which looks like the solution of the harmonic oscillator but in this nonlinear one the frequency depends on the energy or the action.

Chapter 3

Perturbations to Integrable Systems

This chapter includes a very naive outline about the use of canonical transformations in order to deal with small perturbations to integrable systems. In this direction we shall consider a 1D system, $\mathcal{H}_0(p,q)$ which of course is integrable and it can be written as $H_0(I)$, after a canonical change of variables $(p,q) \to (I,\vartheta) \in \mathbb{R} \times \mathbb{S}^1$, where the later are the action–angle variables corresponding to $\mathcal{H}_0$. Clearly we can use either $H_0 = h$ or I as a first integral and just for convenience of this presentation we will take I. Since our system of 1D has one prime integral, we know completely its dynamics.

Now, let us consider a slightly different problem,

$$\mathcal{H}(p,q) = \mathcal{H}_0(p,q) + \varepsilon \mathcal{V}(q), \tag{3.1}$$

where $\varepsilon \ll 1$ is a small positive real number, and $\mathcal{V}(q)$ is a "well behaved" function of the coordinates. It is evident that $\mathcal{H}$ is still integrable, since it is a 1D autonomous system, the *unperturbed* energy h will change by an amount of order ε and so will I, but (I,ϑ) *are not* the action–angle variables of $\mathcal{H}(p,q)$ in a standard sense.

Let us consider $\varepsilon\mathcal{V}(q)$ a *small perturbation* to $\mathcal{H}_0$ and since we know the transformation $(p,q) \to (I,\vartheta)$ we can write

$$H(I,\vartheta) = H_0(I) + \varepsilon V(I,\vartheta), \tag{3.2}$$

and try to solve this trivial 1D system by a perturbation technique. Since I is no longer the action for H, the effect of the perturbation is to introduce a

variation of this prime integral,

$$\dot{I} = -\frac{\partial H}{\partial \vartheta} = -\varepsilon \frac{\partial V}{\partial \vartheta} \neq 0. \tag{3.3}$$

The problem is to find out the motion in H by recourse of our knowledge of the dynamics in H_0. *A Naive Idea*: perform a sequence of canonical transformations,

$$(I, \vartheta) \to (I_1, \vartheta_1) \to \cdots \to (I_n, \vartheta_n) \to (\hat{I}, \hat{\vartheta}), \tag{3.4}$$

such that $\hat{I}$ be a prime integral of H or in other words, $H(I, \vartheta) \to \cdots \to \hat{H}(\hat{I})$. The sequence of canonical transformations should be such that in each step the perturbation decreases in one order,

$$\varepsilon V \to \varepsilon^2 V_1 \to \cdots \to \varepsilon^{n+1} V_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

If we succeed with this approach we would have solved the problem, we need not to integrate the equations of motion.

Let us illustrate this procedure through an example. Let us take the pendulum Hamiltonian and assume that we do not know the existence of the elliptic functions, so we could try a perturbative approach. The full Hamiltonian, for φ not too large, admits the expansion

$$H_p(p, \varphi) = \frac{p^2}{2} - \omega_0^2 \cos\varphi = \frac{p^2}{2} - \omega_0^2 \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \frac{\varphi^8}{8!}\right) + \mathcal{O}(\varphi^{10}).$$

Let us truncate H_p up to $\mathcal{O}(\varphi^8)$, neglect constant terms, set $\omega_0^2 = 1$ and perform the following canonical transformation (polar–like variables):

$$\varphi = \sqrt{2I}\cos\vartheta, \quad p = \sqrt{2I}\sin\vartheta, \qquad I \in \mathbb{R}^+.$$

The new (truncated) Hamiltonian then reads

$$H(I, \vartheta) = I - \frac{4I^2}{4!}\cos^4\vartheta + \frac{8I^3}{6!}\cos^6\vartheta - \frac{16I^4}{8!}\cos^8\vartheta. \tag{3.6}$$

Since the finite expansion of $\cos\varphi$ is valid for relatively small φ, this implies small energies and thus small I, therefore the largest term in (3.6) is the first one and we write

$$\begin{gathered} H(I, \vartheta) = H_0(I) + \varepsilon V(I, \vartheta), \quad \text{where} \quad H_0(I) = I, \quad \text{and} \\ \varepsilon V(I, \vartheta) = -\frac{4I^2}{4!}\cos^4\vartheta + \frac{8I^3}{6!}\cos^6\vartheta - \frac{16I^4}{8!}\cos^8\vartheta. \end{gathered} \tag{3.7}$$

H_0 is the well–known harmonic oscillator Hamiltonian which indeed is linear. The nonlinearity of H comes from high–order terms in I that appear in the perturbation εV. The order of smallness of the perturbation is given by the order of I, it means that $\mathcal{O}(\varepsilon) = \mathcal{O}(I)$, thus H_0 and V are of the same order.

Let us focus our attention to the frequency of the motion in H. Recall that using elliptic functions, up to order h^3, we obtain for the frequency

$$\omega(h) \approx 1 - \frac{h}{8} - \frac{5h^2}{256} - \frac{11h^3}{2048}, \tag{3.8}$$

and let us try to recover it after a perturbative approach. For the sake of comparison, later we will write, up to order $h^3 \sim I^3$, the frequency in terms of the action.

3.1 The easy but inaccurate way

We rewrite each term of the perturbation εV in (3.7), splitting the average value from the oscillatory one like, for instance $\cos^n\vartheta = \langle\cos^n\vartheta\rangle + \eta(\vartheta)$, where $\eta(\vartheta)$ represents the oscillatory part of $\cos^n\vartheta$ and clearly $\langle\eta(\vartheta)\rangle = 0$. Indeed, taking into account that

$$\langle\cos^4\vartheta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^4\vartheta \mathrm{d}\vartheta = \frac{3}{8}, \quad \langle\cos^6\vartheta\rangle = \frac{5}{16}, \quad \langle\cos^8\vartheta\rangle = \frac{35}{128}, \tag{3.9}$$

the perturbation εV could be written in the following way

$$\begin{aligned}\varepsilon V(I,\vartheta) = - & \frac{4}{4!}I^2\left(\cos^4\vartheta - \frac{3}{8}\right) - \frac{4}{4!}\cdot\frac{3}{8}I^2 + \frac{8}{6!}I^3\left(\cos^6\vartheta - \frac{5}{16}\right) + \frac{8}{6!}\cdot\frac{5}{16}I^3 \\ - & \frac{16}{8!}I^4\left(\cos^8\vartheta - \frac{35}{128}\right) - \frac{16}{8!}\cdot\frac{35}{128}I^4.\end{aligned} \tag{3.10}$$

Those terms that only depend on the action correspond to the mean part of $\cos^n\vartheta$, while those depending on both, I and ϑ are the oscillatory ones having zero average respect to the angle variable. Therefore, using the so–called *Averaging technique*[1], that implies neglecting the oscillatory part of the perturbation after averaging the Hamiltonian over the phase ϑ we get from (3.7) and (3.10)

$$\langle H(I,\vartheta)\rangle_\vartheta \approx I - \frac{I^2}{16} + \frac{I^3}{288} - \frac{I^4}{9216}. \tag{3.11}$$

[1]It will be discussed in the next chapter.

The reason to proceed in this way is the (right) idea that the oscillating part of the perturbation only produce *small* vibrations to the secular change. However, as we shall see later this is only valid within certain low accuracy.

3.2 A more rigorous but complicated way

Let us consider the full Hamiltonian (3.7) together with (3.10)

$$H(I,\vartheta) = I - \frac{I^2}{16} + \frac{I^3}{288} - \frac{I^4}{9216} - \frac{I^2}{6}\left(\cos^4\vartheta - \frac{3}{8}\right) + \frac{I^3}{90}\left(\cos^6\vartheta - \frac{5}{16}\right) - \frac{I^4}{2520}\left(\cos^8\vartheta - \frac{35}{128}\right), \tag{3.12}$$

and now, let us introduce a canonical transformation $(I,\vartheta) \to (I_1,\vartheta_1)$ in order to kill terms of order I^2 in the part of the perturbation that depends on the angles. Introducing the generating function

$$F(I_1,\vartheta) = I_1\vartheta + \Phi(I_1,\vartheta) \equiv \mathrm{Id} + \Phi(I_1,\vartheta),$$

such that $\Phi(I_1,\vartheta) \sim \mathcal{O}(I^2)$. The election of Φ should be done in order to eliminate the terms of $\mathcal{O}(I^2)$ in the angles. Recalling the equations to transform canonical variables

$$\begin{aligned} I &= \frac{\partial F}{\partial \vartheta} = I_1 + \frac{\partial \Phi}{\partial \vartheta}, \\ \vartheta_1 &= \frac{\partial F}{\partial I_1} = \vartheta + \frac{\partial \Phi}{\partial I_1}, \end{aligned} \tag{3.13}$$

we observe that $|I - I_1| \sim \mathcal{O}(I^2)$ and $|\vartheta - \vartheta_1| \sim \mathcal{O}(I)$, so that $\mathcal{O}(I^2) \sim \mathcal{O}(I_1^2)$.

The next step is to replace in (3.12), $I \to I_1 + \Phi_\vartheta$ where $\Phi_\vartheta \equiv \partial\Phi/\partial\vartheta$. Later we will deal with the transformation $\vartheta \to \vartheta_1$. The new Hamiltonian then reads

$$H_1(I_1,\vartheta) = I_1 + \Phi_\vartheta - \frac{(I_1+\Phi_\vartheta)^2}{16} + \frac{(I_1+\Phi_\vartheta)^3}{288} - \frac{(I_1+\Phi_\vartheta)^4}{9216} - \frac{(I_1+\Phi_\vartheta)^2}{6}\left(\cos^4\vartheta - \frac{3}{8}\right) + \frac{(I_1+\Phi_\vartheta)^3}{90}\left(\cos^6\vartheta - \frac{5}{16}\right) - \frac{(I_1+\Phi_\vartheta)^4}{2520}\left(\cos^8\vartheta - \frac{35}{128}\right). \tag{3.14}$$

In the above Hamiltonian there are terms of different orders in I_1 and depend on ϑ. Indeed, for instance,

$$\mathcal{O}(I_1^2): \quad \Phi_\vartheta, \, -\frac{I_1^2}{6}\left(\cos^4\vartheta - \frac{3}{8}\right).$$

The rest of the terms that involve the angle are of $\mathcal{O}(I_1^3)$ or higher since they involve products like $I_1^k \Phi_\vartheta^l$ with $k, l \geq 1$ and recall that $\Phi_\vartheta \sim \mathcal{O}(I_1^2)$. So, in order to kill all terms of order $\mathcal{O}(I_1^2)$ that involve the angle it is sufficient to set

$$\Phi_\vartheta - \frac{I_1^2}{6}\left(\cos^4\vartheta - \frac{3}{8}\right) = 0, \quad \text{or} \quad \Phi_\vartheta = \frac{I_1^2}{6}\left(\cos^4\vartheta - \frac{3}{8}\right). \tag{3.15}$$

Note that Φ_ϑ has zero average with respect to ϑ. This is required in order $F(I_1, \vartheta)$ be a well defined generating function of the canonical transformation.

Replacing Φ_ϑ in (3.14), H_1 can be recast, up to $\mathcal{O}(I_1^4)$, as

$$\begin{aligned} H_1(I_1, \vartheta) &= I_1 - \frac{I_1^2}{16} + \frac{I_1^3}{288} - \frac{I_1^4}{9216} - I_1^3\left(\frac{f_4(\vartheta)}{48} - \frac{f_6(\vartheta)}{90} - \frac{f_4^2(\vartheta)}{18}\right) + \\ & I_1^4 \left(\frac{f_4(\vartheta)}{576} - \frac{f_8(\vartheta)}{2520} - \frac{f_4^2(\vartheta)}{576} - \frac{f_4^3(\vartheta)}{216} + \frac{f_4(\vartheta) f_6(\vartheta)}{180}\right), \end{aligned} \tag{3.16}$$

where

$$f_n(\vartheta) = \cos^n\vartheta - \langle\cos^n\vartheta\rangle, \qquad \langle\cos^{2k}\vartheta\rangle = \frac{(2k-1)!!}{(2k)!!}, \qquad \langle\cos^{2k+1}\vartheta\rangle = 0.$$

If we compare (3.16) with (3.11) obtained by the "easy way" we observe that if we average it over the phases we get that both agree up to $\mathcal{O}(I_1^2)$ but at $\mathcal{O}(I_1^3)$ they differ since $\langle f_4^2(\vartheta)\rangle \neq 0$. This simple result provides an estimate of the accuracy of the averaging technique when applied directly to the Hamiltonian written in the original set of canonical variables.

In H_1 the perturbation (that is the terms that depend on ϑ) is smaller but more complicated. Moreover, we should transform $\vartheta \to \vartheta_1$ given by (3.13), $\vartheta_1 = \vartheta + \Phi_{I_1}$. As we have already mentioned, $\Phi_{I_1} \sim \mathcal{O}(I_1)$, thus in order to keep accuracy up to $\mathcal{O}(I_1^4)$ in H_1 we should work on only in the terms at $\mathcal{O}(I_1^3)$, while in those at $\mathcal{O}(I_1^4)$ we simply replace ϑ by ϑ_1. Indeed, since $|\vartheta - \vartheta_1| \sim \mathcal{O}(I_1)$, in the last term in (3.16) the correction is $\mathcal{O}(I_1^5)$, beyond the accuracy we are considering.

Let denote

$$\begin{aligned} Q(\vartheta) &= \frac{f_4(\vartheta)}{48} - \frac{f_6(\vartheta)}{90} - \frac{f_4^2(\vartheta)}{18}, \\ P(\vartheta) &= \frac{f_4(\vartheta)}{576} - \frac{f_8(\vartheta)}{2520} - \frac{f_4^2(\vartheta)}{576} - \frac{f_4^3(\vartheta)}{216} + \frac{f_4(\vartheta)f_6(\vartheta)}{180}, \end{aligned}$$

and expand Q up to first order in ϑ around ϑ_1 and in P just replace ϑ by ϑ_1. Thus we obtain

$$\begin{aligned} Q(\vartheta) &= Q(\vartheta_1) + Q'(\vartheta)|_{\vartheta_1}(\vartheta - \vartheta_1) \quad \text{where} \\ Q'(\vartheta) &= \frac{f_4'}{48} - \frac{f_6'}{90} - \frac{(f_4^2)'}{18}. \end{aligned}$$

By (3.15) the difference in the phases can be computed,

$$\vartheta - \vartheta_1 = -\frac{\partial}{\partial I_1}\int_0^{\vartheta} \frac{I_1^2}{6} f_4(\vartheta')\mathrm{d}\vartheta' = -\frac{I_1}{3}\int_0^{\vartheta} f_4(\vartheta')\mathrm{d}\vartheta' = -\frac{I_1}{3}F_4(\vartheta), \qquad (3.17)$$

where $F_4(\vartheta)$ is the integral of $f_4(\vartheta)$, $F_4(2\pi) = 0$. In the integral (3.17) we can replace as an upper limit ϑ by ϑ_1, since the difference between both phases is of order $\mathcal{O}(I_1)$, and therefore the difference

$$\int_0^{\vartheta} f_4(\vartheta')\mathrm{d}\vartheta' - \int_0^{\vartheta_1} f_4(\vartheta')\mathrm{d}\vartheta' = \mathcal{O}(I_1)$$

and after the product by $I_1/3$ in (3.17), the difference is of order $\mathcal{O}(I_1^2)$. In H_1 since $Q(\vartheta)$ has a factor I_1^3 as it is shown in (3.16), the final correction would be of order $\mathcal{O}(I_1^5)$. Therefore we replace ϑ by ϑ_1 as the upper limit of the integral in order to keep the accuracy up to $\mathcal{O}(I_1^4)$.

Therefore, after transforming to the new variables, the Hamiltonian reads,

$$\begin{aligned} H_1(I_1, \vartheta_1) = I_1 - \frac{I_1^2}{16} + \frac{I_1^3}{288} - \frac{I_1^4}{9216} - I_1^3 Q(\vartheta_1) + \\ \frac{I_1^4}{3}Q'(\vartheta_1)F_4(\vartheta_1) + I_1^4 P(\vartheta_1). \end{aligned} \qquad (3.18)$$

If we now average H_1 over ϑ_1 we need to evaluate

$$\langle Q(\vartheta_1)\rangle, \quad \langle Q'(\vartheta_1)F_4(\vartheta_1)\rangle, \quad \langle P(\vartheta_1)\rangle.$$

Recalling that $\langle f_n(\vartheta)\rangle = 0$, we just need to compute

$$\langle f_4^2(\vartheta)\rangle = \frac{17}{128}, \quad \langle f_4^3(\vartheta)\rangle = \frac{3}{128}, \quad \langle f_4(\vartheta) f_6(\vartheta)\rangle = \frac{33}{256}, \quad \langle f_n'(\vartheta) F_4(\vartheta)\rangle,$$

the last average can be computed by

$$\langle f_n'(\vartheta) F_4(\vartheta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f_n'(\vartheta) F_4(\vartheta)\mathrm{d}\vartheta = -\frac{1}{2\pi}\int_0^{2\pi} f_n(\vartheta) F_4'(\vartheta)\mathrm{d}\vartheta = -\langle f_n(\vartheta) f_4(\vartheta)\rangle$$

where the integration is done by parts and using the fact that $F_4(0) = F_4(2\pi) = 0$. Thus averaging (3.18) and arranging all coefficients results

$$\bar{H}_1(I_1) \equiv \langle H_1(I_1, \vartheta_1)\rangle_{\vartheta_1} = I_1 - \frac{I_1^2}{16} - \frac{I_1^3}{256} - \frac{5I_1^4}{2^{13}}. \tag{3.19}$$

If we compare (3.19) with (3.11) we observe that both Hamiltonians agree up to order I^2 but differ for higher orders and thus the frequency do up to order I. In this particular case we get

$$\omega(I_1) = \frac{\partial \bar{H}_1(I_1)}{\partial I_1} = 1 - \frac{I_1}{8} - \frac{3I_1^2}{256} - \frac{5I_1^3}{2^{11}}. \tag{3.20}$$

In order to compare this estimation of the frequency with that obtained for the pendulum using elliptic functions, let us write (3.20) in terms of the energy, h, just setting $\bar{H}_1(I_1) = h$ and consider $I_1 = h + \xi$, with $\xi \sim \mathcal{O}(h^3) \ll 1$. Introducing this expression for I_1 in (3.19), up to h^3 we get

$$h = h + \xi - \frac{(h+\xi)^2}{16} - \frac{(h+\xi)^3}{256} + \mathcal{O}(h^4),$$

and keeping only the linear terms in ξ and recalling that $\mathcal{O}(h^2\xi) = \mathcal{O}(h\xi^2) = \mathcal{O}(h^4)$ (or higher) it reduces to

$$0 = \xi - \frac{h^2}{16} - \frac{h\xi}{8} - \frac{h^3}{256}, \qquad \to \qquad \xi \approx \frac{h^2}{16} + \frac{3h^3}{256}.$$

Therefore, the action, up to $\mathcal{O}(h^3)$, is

$$I_1 \approx h + \frac{h^2}{16} + \frac{3h^3}{256}$$

and introducing it in (3.20) up to h^3 results,

$$\omega(h) \approx 1 - \frac{h}{8} - \frac{5h^2}{256} - \frac{11h^3}{2048}, \tag{3.21}$$

and we reobtain the very same expression for $\omega(h)$, up to h^3, for the simple pendulum given in (3.8).

3.3 Kolmogorov Superconvergence

If we keep H_1 given in (3.18) without averaging, we could introduce a new canonical transformation in order to kill those terms in H_1 that depend on the phase ϑ_1 at $\mathcal{O}(I_1^3)$. Let

$$G(I_2, \vartheta_1) = I_2\vartheta_1 + \Psi(I_2, \vartheta_1) \equiv \mathrm{Id} + \Psi(I_2, \vartheta_1),$$

be the generating function where $\Psi(I_2, \vartheta_1)$ is, by now an unknown function. The relation between old and new variables is

$$I_1 = I_2 + \Psi_{\vartheta_1}, \qquad \vartheta_2 = \vartheta_1 + \Psi_{I_2}.$$

Now $\Psi(I_2, \vartheta_1) \sim \mathcal{O}(I_1^3) \sim \mathcal{O}(I_2^3)$. Let us rewrite H_1 given by (3.18) as

$$H_1(I_1, \vartheta_1) = I_1 - \frac{I_1^2}{16} + a_1 I_1^3 - a_2 I_1^4 - g_1(\vartheta_1) I_1^3 + I_1^4 g_2(\vartheta_1), \tag{3.22}$$

where the average of the terms involving ϑ_1 in I_1^3 and I_1^4 are incorporated in the numerical constants $a_1$and a_2 so that $\langle g_i(\vartheta_1)\rangle = 0$. Indeed, as we proceed before, we separate the average and mean values of all the terms that depend on ϑ_1, so $a_1 = -1/256, a_2 = 5/2^{11}$. Following the same procedure, we replace $I_1 = I_2 + \Psi_{\vartheta_1}$ in (3.22),

$$\begin{aligned} H_2(I_2, \vartheta_1) = I_2 + \Psi_{\vartheta_1} - \frac{(I_2 + \Psi_{\vartheta_1})^2}{16} + a_1(I_2 + \Psi_{\vartheta_1})^3 - \\ a_2(I_2 + \Psi_{\vartheta_1})^4 - g_1(\vartheta_1)(I_2 + \Psi_{\vartheta_1})^3 + (I_2 + \Psi_{\vartheta_1})^4 g_2(\vartheta_1). \end{aligned} \tag{3.23}$$

Since Ψ_{ϑ_1} is $\mathcal{O}(I_2^3)$ we can write

$$\Psi_{\vartheta_1} = I_2^3 h(\vartheta_1), \qquad \langle h(\vartheta_1)\rangle = 0,$$

where Ψ_{ϑ_1} depends on ϑ_1 through the oscillating function $h(\vartheta_1)$. Therefore, up to $\mathcal{O}(I_2^5)$, (3.23) reads

$$\begin{aligned} H_2(I_2, \vartheta_1) = I_2 - \frac{I_2^2}{16} + a_1 I_1^3 - a_2 I_1^4 + I_2^3 h(\vartheta_1) - \\ \frac{1}{8} I_2^4 h(\vartheta_1) - g_1(\vartheta_1) I_2^3 + g_2(\vartheta_1) I_2^4. \end{aligned} \tag{3.24}$$

From the above expression if we take $h(\vartheta_1) = g_1(\vartheta_1)$ we succeed in killing the terms in the perturbation at $\mathcal{O}(I_2^3)$ and H_2 reduces to

$$H_2(I_2,\vartheta_2) = I_2 - \frac{I_2}{16} + a_1 I_1^3 - a_2 I_1^4 - \frac{1}{8} I_2^4 g_1(\vartheta_2) + g_2(\vartheta_2) I_2^4, \qquad (3.25)$$

where we have replaced ϑ_1 by ϑ_2, since $\vartheta_2 - \vartheta_1 \sim \mathcal{O}(I_2^2)$, and taken into account that the difference $|g_i(\vartheta_2) - g_i(\vartheta_1)| \sim \mathcal{O}(I_2^2)$ and after the product by I_2^4 in (3.25), they lead to terms of $\mathcal{O}(I_2^6)$, beyond of the desired accuracy.

Though we have already killed all terms in the angles at $\mathcal{O}(I_2^3)$, we observe from (3.25) that, since $\langle g_1\rangle = \langle g_2\rangle = 0$, if we average H_2 we are eliminating simultaneously terms in ϑ_2 at order I_2^3 and I_2^4 leading to

$$\langle H_2\rangle_{\vartheta_2} = I_2 - \frac{I_2}{16} + a_1 I_1^3 - a_2 I_1^4. \qquad (3.26)$$

If we repeat this procedure, after a new canonical transformation, we should find that in H_3 the perturbation (the terms that depend on the angles) is $\mathcal{O}(I_3^9)$. In general, in the n-th canonical transformation the perturbation is $V_n \sim \mathcal{O}(I_n^{k_n})$, therefore in the next step, $V_{n+1} \sim \mathcal{O}(I_2^{2k_n})$. Indeed, if we rename Φ by $\Phi^{(0)}$ and Ψ by $\Phi^{(1)}$, in the first step we obtain $V_1 \sim I_1\Phi^{(0)} \sim \mathcal{O}(I_1^3)$, in the second one $V_2 \sim I_2^2\Phi^{(1)} \sim \mathcal{O}(I_2^5)\ldots$, thus $k_n = 2^n + 1$. This is known as *Kolmogorov superconvergence* that allows to construct convergent series. Indeed, as we have already mentioned, after a canonical transformation, the perturbation becomes smaller but more complicated. Therefore the *remainder* or n-th perturbation takes the form

$$V_n(I_n,\vartheta_n) = \sum_{l=0}^{\infty} A_l(\vartheta_n) I_n^{k_n+l},$$

where the $A_l(\vartheta_n)$ depends on all the $\Phi^{(m)}, m = 0,\ldots,n-1$, while the mean Hamiltonian, that only depends on the actions, is

$$\bar{H}_n(I_n) = \sum_{s=1}^{k_{n-1}} \alpha_s I_n^s,$$

where the α_s are numerical coefficients, with $\alpha_1 = 1, \alpha_2 = -1/16, \alpha_3 = a_1 = -1/256, \ldots$.

One should keep in mind that the convergence of the series (which in a generic N-dimensional problem, $N > 1$, are divergent) is due to the fact

that we are dealing with an integrable problem, where any perturbation to H_0 produces a slight change in the action, and therefore its effect is to move from one torus to another. In other words, we start on a given torus in H_0 and the effect of switching on the perturbation is to end in a slightly different torus. This is not the case if $H_0 + \epsilon V$ is non–integrable, which is the most probable case in the real world. However, this approach is very useful in, for instance, the so–called KAM theory.

Chapter 4

Averaging

4.1 The principle of averaging

Let $(\boldsymbol{I}, \boldsymbol{\vartheta})$ be action-angle variables of an integrable Hamiltonian system (unperturbed model), $H_0(\boldsymbol{I})$, then

$$\dot{\boldsymbol{I}} = 0, \qquad \dot{\boldsymbol{\vartheta}} = \boldsymbol{\omega}(\boldsymbol{I}) = \frac{\partial H_0}{\partial \boldsymbol{I}}.$$

As a perturbed (close to integrable) system we take

$$\dot{\boldsymbol{I}} = \varepsilon \boldsymbol{g}(\boldsymbol{I}, \boldsymbol{\vartheta}), \qquad \dot{\boldsymbol{\vartheta}} = \boldsymbol{\omega}(\boldsymbol{I}) + \varepsilon \boldsymbol{f}(\boldsymbol{I}, \boldsymbol{\vartheta}), \qquad \varepsilon \ll 1, \tag{4.1}$$

where $\boldsymbol{g}$ and $\boldsymbol{f}$ are periodic in $\boldsymbol{\vartheta}$ of period 2π. For the time being, let us ignore that we are dealing with a Hamiltonian system and consider an arbitrary system of differential equations of the form (4.1), defined in $S^k \times G$, from the k-dimensional torus, $S^k = \{\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_k) \bmod 2\pi\}$ and the region G in the l-dimensional manifold $G = \{\boldsymbol{I} = (I_1, \ldots, I_l)\} \subset \mathbb{R}^l$. We assume that for $\varepsilon = 0$ the motion is quasiperiodic with at least k frequencies and invariant tori of dimension k.

The *averaging principle* for the system (4.1) consists in replacing it by the the so-called mean or averaged system

$$\dot{\boldsymbol{J}} = \varepsilon \bar{\boldsymbol{g}}(\boldsymbol{J}), \qquad \bar{\boldsymbol{g}}(\boldsymbol{J}) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \mathrm{d}\vartheta_1 \int_0^{2\pi} \mathrm{d}\vartheta_2 \cdots \int_0^{2\pi} \mathrm{d}\vartheta_k \boldsymbol{g}(\boldsymbol{J}, \boldsymbol{\vartheta}), \tag{4.2}$$

in the l-dimensional manifold $G = \{\boldsymbol{J} = (J_1, \ldots, J_l)\}$.

We affirm that the system (4.2) is a "good approximation" to the system (4.1). Let us state that this *principle* is not a theorem, nor an axiom or a definition. *It is a physical proposition*, vaguely formulated and, strictly speaking, false. Such kind of physical affirmations are frequently a rich source of mathematical theorems. In fact, the averaging principle could be explicitly found in the early work of Gauss, however a satisfactory proof of the connections of the solutions of systems (4.1) and (4.2) is still lacking.

Replacing system (4.1) by (4.2) implies that we are neglecting the term

$$\varepsilon\tilde{\boldsymbol{g}}(\boldsymbol{J},\boldsymbol{\vartheta}) = \varepsilon\boldsymbol{g}(\boldsymbol{J},\boldsymbol{\vartheta}) - \varepsilon\bar{\boldsymbol{g}}(\boldsymbol{J})$$

in the right hand side in (4.1).

In order to understand the different roles of the terms $\bar{\boldsymbol{g}}$ and $\tilde{\boldsymbol{g}}$ in $\boldsymbol{g}$ let us consider the simplest case, $k = l = 1$,

$$\begin{gathered} \dot{I} = \varepsilon g(\vartheta), \qquad \dot{\vartheta} = \omega > 0, \quad \to \quad \vartheta(t) = \omega t + \vartheta_0, \\ \dot{J} = \varepsilon\bar{g}, \qquad \text{where} \qquad g(\vartheta) = \bar{g} + \tilde{g}(\vartheta), \end{gathered} \tag{4.3}$$

and we will show that for[1] $0 < t < 1/\varepsilon$

$$|I(t) - J(t)| < c\varepsilon, \quad c > 0, \quad \text{where} \quad J(t) = I(0) + \varepsilon\bar{g}t.$$

Let us compute $I(t) - I(0)$. From the first of (4.3) we have

$$I(t) - I(0) = \int_0^t \varepsilon g(\omega t' + \vartheta_0)\mathrm{d}t' = \int_0^t \varepsilon\bar{g}\mathrm{d}t' + \int_0^t \varepsilon\tilde{g}(\omega t' + \vartheta_0)\mathrm{d}t',$$

let $\vartheta = \omega t + \vartheta_0$, then $\mathrm{d}\vartheta = \omega\mathrm{d}t$, thus

$$I(t) - I(0) = \varepsilon\bar{g}t + \frac{\varepsilon}{\omega}\int_{\vartheta_0}^{\vartheta_0+\omega t} \tilde{g}(\vartheta)\mathrm{d}\vartheta = \varepsilon\bar{g}t + \frac{\varepsilon}{\omega}h(\omega t),$$

where

$$h(\vartheta) = \int_{\vartheta_0}^{\vartheta} \tilde{g}(\vartheta')\mathrm{d}\vartheta'$$

is periodic and therefore bounded.

Therefore the variation of I with time consists of two parts, an oscillation of order ε due to $\tilde{g}$ and a *secular evolution* with a speed $\varepsilon\bar{g}$.

[1] Later we will discuss this upper bound for the time.

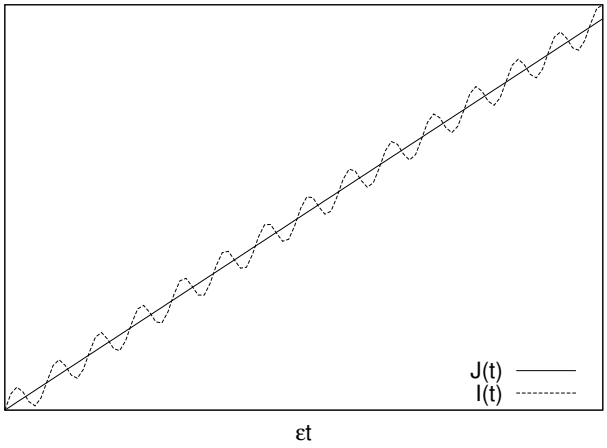

Figure 4.1: Schematic representation of the time evolution of $I(t)$ and $J(t)$. The solid line corresponds to $J(t)$ while the dotted one to $I(t)$.

Finally,

$$|I(t) - J(t)| = |\varepsilon\bar{g}t + \frac{\varepsilon}{\omega}h(\omega t) + I(0) - I(0) - \varepsilon\bar{g}t| = \frac{\varepsilon}{\omega}|h(\omega t)| < c\varepsilon.$$

Fig.4.1 represents schematically the evolution of I and J with time. The averaging principle is based on the assertion that, in general, the motion in the system (4.1) can be splitted in to parts: the *evolution* (4.2) and small oscillations. In the general form, this assertion is not true and the principle is false. However, we will apply it to the Hamiltonian system (4.1),

$$\dot{\boldsymbol{\vartheta}} = \frac{\partial}{\partial \boldsymbol{I}}\left(H_0(\boldsymbol{I}) + \varepsilon V(\boldsymbol{I}, \boldsymbol{\vartheta})\right), \quad \dot{\boldsymbol{I}} = -\frac{\partial}{\partial \boldsymbol{\vartheta}}\left(H_0(\boldsymbol{I}) + \varepsilon V(\boldsymbol{I}, \boldsymbol{\vartheta})\right),$$

where

$$\boldsymbol{f}(\boldsymbol{I}, \boldsymbol{\vartheta}) = \frac{\partial V(\boldsymbol{I}, \boldsymbol{\vartheta})}{\partial \boldsymbol{I}}, \quad \boldsymbol{g}(\boldsymbol{I}, \boldsymbol{\vartheta}) = -\frac{\partial V(\boldsymbol{I}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}}, \tag{4.4}$$

with $V(\boldsymbol{I}, \boldsymbol{\vartheta})$ 2π-periodic in $\vartheta_1, \vartheta_2, \ldots, \vartheta_k$. For the system (4.2) we get

$$\dot{\boldsymbol{J}} = \varepsilon\bar{\boldsymbol{g}}(\boldsymbol{J}) = -\frac{1}{(2\pi)^k}\int_{S^k} \frac{\partial V(\boldsymbol{I}, \boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}} \mathrm{d}\boldsymbol{\vartheta} = 0$$

due to the periodicity of V. Therefore, there is no evolution in a (non-degenerated) Hamiltonian system.

Let us discuss this point more carefully. Since both $\boldsymbol{g}$ and V are assumed to be periodic in all the phases, we can expand them in Fourier series

$$\boldsymbol{g}(\boldsymbol{I},\boldsymbol{\vartheta}) = \boldsymbol{g}_0(\boldsymbol{I}) + \sum_{\boldsymbol{m}\neq 0} \boldsymbol{g}_{\boldsymbol{m}}(\boldsymbol{I}) e^{i\boldsymbol{m}\cdot\boldsymbol{\vartheta}}, \quad V(\boldsymbol{I},\boldsymbol{\vartheta}) = V_0(\boldsymbol{I}) + \sum_{\boldsymbol{m}\neq 0} V_{\boldsymbol{m}}(\boldsymbol{I}) e^{i\boldsymbol{m}\cdot\boldsymbol{\vartheta}},$$

where $\boldsymbol{m} \in \mathbb{Z}^k$, and $\boldsymbol{g}_{\boldsymbol{m}}$ and $V_{\boldsymbol{m}}$ are complex coefficients that depend on the action. Thus in case of a Hamiltonian system, since $\boldsymbol{g}$ and V are related accordingly to (4.4), it should be

$$\boldsymbol{g}_0(\boldsymbol{I}) = \bar{\boldsymbol{g}}(\boldsymbol{I}) = 0.$$

However in the general case of a system of differential equations, where (4.4) does not apply, $\boldsymbol{g}_0(\boldsymbol{I}) \neq 0$, and therefore a secular change of the action with time is present.

4.2 Averaging a system of a single frequency

Let us consider $k = 1$ and then the $l + 1$ system of differential equations

$$\dot{\boldsymbol{I}} = \varepsilon \boldsymbol{g}(\boldsymbol{I},\vartheta), \qquad \dot{\vartheta} = \omega(\boldsymbol{I}) + \varepsilon f(\boldsymbol{I},\vartheta), \quad \vartheta \in S^1, \quad \boldsymbol{I} \in G \subset \mathbb{R}^l. \tag{4.5}$$

Assume that both $\boldsymbol{g}$ and f are 2π-periodic in ϑ; and consider the averaged system of l differential equations

$$\dot{\boldsymbol{J}} = \varepsilon \bar{\boldsymbol{g}}(\boldsymbol{J}), \qquad \bar{\boldsymbol{g}}(\boldsymbol{J}) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{g}(\boldsymbol{J},\vartheta)\mathrm{d}\vartheta. \tag{4.6}$$

Let $\boldsymbol{I}(t), \vartheta(t)$ be a given solution of (4.5) with initial condition $\boldsymbol{I}(0), \vartheta(0)$ and $\boldsymbol{J}(t)$ a solution of (4.6) with the same initial condition, $\boldsymbol{J}(0) = \boldsymbol{I}(0)$.

Theorem

Assume that

i) The functions ω, f and $\boldsymbol{g}$ are defined for $\boldsymbol{I}$ in a bounded region of G and in this region they are bounded as well as their derivatives up to second order;

ii) In G, $\omega > c > 0$;

For $0 < t < 1/\varepsilon$ a vicinity of radius d of the point $\boldsymbol{J}(t)$ belongs to G, then for ε small enough

$$|\boldsymbol{I}(t) - \boldsymbol{J}(t)| < c_1\varepsilon,$$

where c_1 is a positive constant independent of ε.

Proof (non-rigorous)[2]

Let us introduce a new variable

$$\boldsymbol{P} = \boldsymbol{I} + \varepsilon \boldsymbol{K}(\boldsymbol{I}, \vartheta),$$

where $\boldsymbol{K}$ is 2π-periodic in ϑ. The function $\boldsymbol{K}$ will be such that, in terms of the variable $\boldsymbol{P}$, the corresponding differential equation becomes simpler than that for $\boldsymbol{I}$. From its definition, we have

$$\dot{\boldsymbol{P}} = \dot{\boldsymbol{I}} + \varepsilon \dot{\boldsymbol{K}}(\boldsymbol{I}, \vartheta) = \dot{\boldsymbol{I}} + \varepsilon \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{I}} \dot{\boldsymbol{I}} + \varepsilon \frac{\partial \boldsymbol{K}}{\partial \vartheta} \dot{\vartheta},$$

replacing $\dot{\boldsymbol{I}}$ and $\dot{\vartheta}$ using (4.5), up to $\mathcal{O}(\varepsilon)$, we get

$$\dot{\boldsymbol{P}} = \varepsilon \left(\boldsymbol{g}(\boldsymbol{I}, \vartheta) + \omega(\boldsymbol{I}) \frac{\partial \boldsymbol{K}}{\partial \vartheta} \right) + \mathcal{O}(\varepsilon^2).$$

Assuming that the function $\boldsymbol{P}(\boldsymbol{I}, \vartheta)$ is invertible[3] with respect to $\boldsymbol{I}$ such that

$$\boldsymbol{I} = \boldsymbol{P} + \varepsilon \boldsymbol{h}(\boldsymbol{P}, \vartheta),$$

with $\boldsymbol{h}$, 2π-periodic in ϑ, then

$$\dot{\boldsymbol{P}} = \varepsilon \left(\boldsymbol{g}(\boldsymbol{P} + \varepsilon \boldsymbol{h}, \vartheta) + \omega(\boldsymbol{P} + \varepsilon \boldsymbol{h}, \vartheta) \frac{\partial \boldsymbol{K}}{\partial \vartheta} (\boldsymbol{P} + \varepsilon \boldsymbol{h}, \vartheta) \right) + \mathcal{O}(\varepsilon^2),$$

which up to $\mathcal{O}(\varepsilon)$ reduces to

$$\dot{\boldsymbol{P}} = \varepsilon \left(\boldsymbol{g}(\boldsymbol{P}, \vartheta) + \omega(\boldsymbol{P}) \frac{\partial \boldsymbol{K}}{\partial \vartheta} (\boldsymbol{P}, \vartheta) \right) + \boldsymbol{R}, \tag{4.7}$$

where the remainder $\boldsymbol{R} \sim \mathcal{O}(\varepsilon^2)$. If we choose K such that

$$\boldsymbol{g}(\boldsymbol{P}, \vartheta) + \omega(\boldsymbol{P}) \frac{\partial \boldsymbol{K}}{\partial \vartheta} (\boldsymbol{P}, \vartheta) = 0,$$

[2]See Arnold, *Mathematical Methods of Classical Mechanics*, Springer, 1989.

[3]Although this can be proved, recall that it is possible to perform a canonical transformation where the old action is a function of the new one.

we should eliminate all the terms of $\mathcal{O}(\varepsilon)$, and then

$$\frac{\partial \boldsymbol{K}}{\partial \vartheta} = -\frac{\boldsymbol{g}(\boldsymbol{P},\vartheta)}{\omega(\boldsymbol{P})}. \tag{4.8}$$

This equation does not have solution within the class of periodic functions $\boldsymbol{K}$, since for instance, if we average over the angle the above equation we get that

$$\langle \frac{\partial \boldsymbol{K}}{\partial \vartheta} \rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \boldsymbol{K}}{\partial \vartheta} \mathrm{d}\vartheta = 0,$$

while in the general case

$$\langle \boldsymbol{g}(\boldsymbol{P},\vartheta)\rangle \equiv \bar{\boldsymbol{g}}(\boldsymbol{P}) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{g}(\boldsymbol{P},\vartheta)\mathrm{d}\vartheta \neq 0.$$

Therefore we cannot choose $\boldsymbol{K}$ such that it kills all the terms of $\mathcal{O}(\varepsilon)$. However if we separate the mean and oscillating terms of $\boldsymbol{g}$ such that

$$\tilde{\boldsymbol{g}}(\boldsymbol{P},\vartheta) = \boldsymbol{g}(\boldsymbol{P},\vartheta) - \bar{\boldsymbol{g}}(\boldsymbol{P})$$

with $\langle \tilde{\boldsymbol{g}} \rangle = 0$ we can eliminate all the periodic terms at order $\mathcal{O}(\varepsilon)$. So if in (4.8) we replace $\boldsymbol{g}$ by $\tilde{\boldsymbol{g}}$ and defining $\boldsymbol{K}$ as

$$\boldsymbol{K}(\boldsymbol{P},\vartheta) = -\frac{1}{\omega(\boldsymbol{P})}\int_0^{\vartheta} \tilde{\boldsymbol{g}}(\boldsymbol{P},\vartheta')\mathrm{d}\vartheta',$$

if we replace in (4.7), it reads

$$\begin{aligned}
\dot{\boldsymbol{P}} &= \varepsilon\left(\bar{\boldsymbol{g}}(\boldsymbol{P}) + \tilde{\boldsymbol{g}}(\boldsymbol{P},\vartheta) + \omega(\boldsymbol{P})\left(\frac{-\tilde{g}(\boldsymbol{P},\vartheta)}{\omega(\boldsymbol{P})}\right)\right) + \boldsymbol{R}\left(\mathcal{O}(\varepsilon^2)\right) \\
&= \varepsilon\bar{\boldsymbol{g}}(\boldsymbol{P}) + \boldsymbol{R}\left(\mathcal{O}(\varepsilon^2)\right).
\end{aligned}$$

Since by (4.6), the average system is $\dot{\boldsymbol{J}} = \varepsilon\bar{\boldsymbol{g}}(\boldsymbol{J})$ we observe that the difference between both systems if of order $\mathcal{O}(\varepsilon^2)$, and therefore for $0 < t \lesssim 1/\varepsilon$ it is $|\boldsymbol{P}(t) - \boldsymbol{J}(t)| \sim \mathcal{O}(\varepsilon)$,[4] and since $|\boldsymbol{I}(t) - \boldsymbol{P}(t)| \sim \mathcal{O}(\varepsilon)$, then $|\boldsymbol{I}(t) - \boldsymbol{J}(t)| \sim \mathcal{O}(\varepsilon)$. The remainder $\boldsymbol{R}$, contains at $\mathcal{O}(\varepsilon^2)$, the values of $f, \omega, \boldsymbol{g}$ and their derivatives, thus it is necessary that all of them are bounded, as it was stated as assumptions of this theorem.

[4] Clearly if $|\dot{\boldsymbol{J}} - \dot{\boldsymbol{P}}| \sim \mathcal{O}(\varepsilon^2)$, then for $t \lesssim 1/\varepsilon$ the difference between $\boldsymbol{J}$ and $\boldsymbol{P}$ is of $\mathcal{O}(\varepsilon)$.

Summing up we have shown that the averaging principle allow us to eliminate all terms that depend on the phase up to $\mathcal{O}(\varepsilon)$. However as we shall show in the next Chapter by means of a simple example, it is not possible to do this if the system involves more than a single frequency. This is the reason why this principle, though very useful, it is not true in general.

Chapter 5

Non-linear resonance

Let us consider a 1D Hamiltonian system acted upon an external perturbation

$$H(I, \vartheta, t) = H_0(I) + \varepsilon V(I, \vartheta, t), \qquad \varepsilon \ll 1,$$

where $I \in G \subset \mathbb{R}, \vartheta \in S^1$ are the action-angle variables for H_0. We assume that $V(I, \vartheta, t)$ is also periodic in time with period T

$$V(I, \vartheta, t + T) = V(I, \vartheta, t),$$

so we can introduce the external frequency, Ω, and phase, τ, through

$$\Omega = \frac{2\pi}{T}, \qquad \tau(t) = \Omega t + \tau_0$$

such that, thus defined $\tau \in S^1$ (or mod 2π). Therefore the dependence of V on t is through $\Omega t + \tau_0$. Since $V(I, \vartheta, t) \equiv V(I, \vartheta, \tau)$ is periodic in both phases, it admits a Fourier expansion

$$V(I, \vartheta, \tau) = \sum_{m,n \neq (0,0)} V_{mn}(I) e^{i(m\vartheta + n\tau)},$$

where $m, n \in \mathbb{Z}, V_{mn}(I)$ are complex functions of the action and we have assumed that $V_{00}(I) = 0$. Therefore, the Hamiltonian takes the form

$$H(I, \vartheta, \tau) = H_0(I) + \varepsilon \sum_{m,n \neq (0,0)} V_{mn}(I) e^{i(m\vartheta + n\tau)}. \tag{5.1}$$

If instead, $V_{00}(I) \neq 0$, we can add this term to $H_0(I)$ and we get $H_0'(I, \varepsilon)$ and we call it again $H_0(I)$.

It is clear that for $\varepsilon = 0, I$ (or the energy $H_0(I) = h$) is a global integral of motion. From the Hamilton equations, for $\varepsilon \neq 0$, the variation of I is given by

$$\dot{I} = -\frac{\partial H}{\partial \vartheta} = -\varepsilon \sum_{m,n\neq 0} V_{mn}(I) im e^{i(m\vartheta + n\tau)} = \varepsilon \sum_{m,n\neq(0,0)} \hat{V}_{mn}(I) e^{i(m\vartheta+n\tau)},$$

after introducing $\hat{V}_{mn}(I) = -V_{mn}(I)im$. Let us assume that for a given initial condition and particular values of $m, n = m_0, n_0$ it happens that

$$m_0\vartheta + n_0\tau \equiv \psi_{00} \approx \text{constant},$$

nearly independent of the time. Therefore, we separate this term from the sum and we write

$$\dot{I} = \underbrace{\varepsilon \hat{V}_{m_0 n_0}(I) e^{i\psi_{00}}}_{\text{independent of } \vartheta \text{ and } \tau} + \varepsilon \sum_{m,n\neq m_0,n_0} \hat{V}_{mn}(I) e^{i(m\vartheta+n\tau)}.$$

As we have already seen when discussing averaging, the first term that is nearly independent of the angles, would lead to a secular growth of $I(t)$ (it is similar to the average system), and therefore there is evolution in the average system. Thus, the dominant term in the above Fourier series would be that satisfying

$$|\dot{\psi}_{mn}| \equiv |m\dot{\vartheta} + n\dot{\tau}| \ll 1. \tag{5.2}$$

Clearly, if this condition holds for some pair m, n, then the term with km, kn, $k \in \mathbb{Z}$ also satisfies this condition. However, since the coefficients of the Fourier series decay very fast, $|\hat{V}_{(km)(kn)}| \ll |\hat{V}_{mn}|$, $|k| > 1$, in this first approximation we shall consider only the smallest m, n such that $|\dot{\psi}_{mn}| \ll 1$.

In the limit when $\dot{\psi}_{mn} = m\dot{\vartheta} + n\dot{\tau} = 0$, (5.2) leads to the resonance condition

$$m\dot{\vartheta} + n\dot{\tau} = 0 \qquad \Rightarrow \qquad m\omega_0(I) + n\Omega = 0, \tag{5.3}$$

where we take for $\dot{\vartheta} = \partial H_0/\partial I = \omega_0$ the unperturbed frequency. The above relation (5.3) is the resonance condition for the frequency of the unperturbed motion, $\omega_0(I)$, with the external frequency Ω for $m, n \neq 0$. Let us denote this frequency value ω_{0r}. Alternatively, this is a condition on the unperturbed resonant action I_r, that corresponds to the action value that satisfies (5.3), with $\omega_0(I_r) = \omega_{0r}$. Moreover, since $H_0(I) = h$, we can also refer to the resonant energy h_r, defined as $H_0(I_r) = h_r$. Note that since we are considering

non-linear systems, the resonance condition is *local* in action or energy space. Indeed, in a linear system, since the frequency is independent of the action or the energy, if the system is in resonance, it will be resonant for any value of the action or the energy.

For example, consider an harmonic oscillator, $H_0(I) = \omega_l I$, where ω_l is the (constant) linear frequency. Thus the resonance condition for $\omega_0 = \partial H_0/\partial I = \omega_l$, reads $m\omega_l + n\Omega = 0$, independent of the action or energy value, and also of the initial conditions.

Instead, let us take a quartic oscillator, $H_0(I) = AI^{4/3}$, where A is a numerical constant. The frequency for this nonlinear oscillator is $\omega_0(I) = (4/3)AI^{1/3}$, and thus the resonance condition becomes $4mAI_r^{1/3} + 3n\Omega = 0$. This leads to a single value of the resonant action and therefore for the energy, h_r (for a given pair of m, n), at which the non-linear oscillator is in resonance with the external perturbation. This fact is of major relevance, as we shall see along this chapter.

In general, for all pairs of integers $(m, n) \in \mathbb{Z}^2/\{0\}$, the set $\omega_{mn} = -n/m\Omega$ is dense in phase space, since $\omega_{mn}/\Omega \in \mathbb{Q}$, the latter being dense in $\mathbb{R}$. Therefore, for a given perturbation of frequency Ω, there exists an infinite (but numerable) set of resonant values of the frequency, action or energy that satisfy the resonance condition.

5.1 An illustrative example

Let us consider (5.1), take just the real part of the Fourier series and instead of a infinite set or harmonics, assume that only two are present, such that

$$H(I, \vartheta, \tau) = H_0(I) + \varepsilon V_1(I)\cos(m\vartheta - n\tau) + \varepsilon V_2(I)\cos(r\vartheta + s\tau), \qquad (5.4)$$

where m, n, r, s are positive integer numbers, V_i are real functions of the action and assume that the external frequency $\Omega > 0$. Take for instance some energy level, $h = h_*$, for the unperturbed system within the interval (h_a, h_b), then $H_0(I_*) = h_*$ for $I_* \in (I_a, I_b)$. The energy level fix the initial condition for the action value, which for the unperturbed motion is constant, $I(t) = I_*, \forall t$. Assume that for $I_* \in (I_a, I_b)$, $\omega_0(I_*) = (\partial H_0/\partial I)_{I_*} > 0$.

The time evolution of the phases is then, $\vartheta(t) = \omega_0(I_*)t + \vartheta_0$ and $\tau(t) = \Omega t + \tau_0$ and let us take the rest of initial conditions such that $\vartheta_0 = \tau_0 = 0$. It is clear then that argument of the second term of the perturbation $\phi_f \equiv (r\vartheta(t) + s\tau(t)) = (r\omega_0(I_*) + s\Omega)t$ is always positive and *fast*, since its

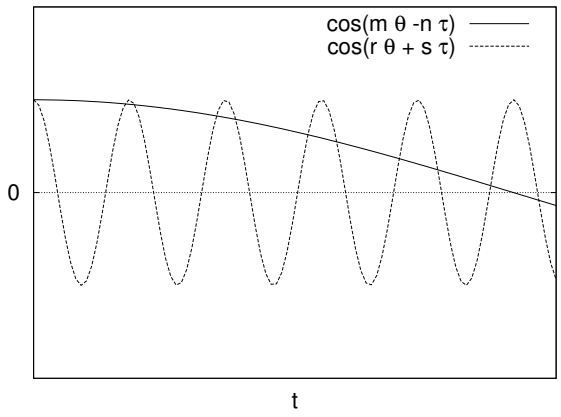

Figure 5.1: Schematic representation of the evolution of the slow and fast phases over a given time span.

frequency $r\omega_0(I_*) + s\Omega$ is larger than $\omega_0(I_*)$ or Ω. On the other hand, the argument of the other term $\phi_s = (m\vartheta(t) - s\tau(t)) = (m\omega_0(I_*) - n\Omega)t$ could be positive or negative for a given pair of integers $m, n > 0$ and then it is *slow*, its frequency $m\omega_0(I_*) - n\Omega$ could be zero for a given value of $I_* \in (I_a, I_b)$.

Fig. 5.1 represents schematically this situation for both phases. After a given time span, the fast phase presents several oscillations with zero average while in the same time interval, the slow phase does not exhibit any oscillation.

Therefore for $|m\omega_0(I_*) - n\Omega| \ll 1$ for $I_* \in (I_a, I_b)$, $r\omega_0(I_*) + s\Omega$ is large and we can average (5.4) over the fast phase ϕ_f and we get

$$\langle H(I, \vartheta, \tau)\rangle_{\phi_f} = H_0(I) + \varepsilon V_1(I)\cos(m\vartheta - n\tau), \qquad |m\omega_0(I_*) - n\Omega| \ll 1. \quad (5.5)$$

Now let us follow the technique developed in the previous chapters: perform a canonical transformation $(I, \vartheta) \to (I_1, \vartheta_1)$ in order to kill the terms of $\mathcal{O}(\varepsilon)$ in the Hamiltonian (5.5), assuming $I, I_1 \in (I_a, I_b)$. To this end, we introduce the generating function

$$F(I_1, \vartheta, \tau) = I_1\vartheta + \varepsilon\Phi(I_1)\sin(m\vartheta - n\tau),$$

where $\Phi(I_1)$ is by now an unknown function of the action. Recalling the

relation between old and new variables

$$\begin{aligned} I &= \frac{\partial F}{\partial \vartheta} = I_1 + \varepsilon m \Phi(I_1)\cos(m\vartheta - n\tau), \\ \vartheta_1 &= \frac{\partial F}{\partial I_1} = \vartheta + \Phi'(I_1)\sin(m\vartheta - n\tau), \\ \tilde{H} &= \langle H\rangle_{\phi_f} + \frac{\partial F}{\partial t} = \langle H\rangle_{\phi_f} + \frac{\partial F}{\partial \tau}\underbrace{\frac{\mathrm{d}\tau}{\mathrm{d}t}}_{\Omega} \\ &= H_0(I) + \varepsilon V_1(I)\cos(m\vartheta - n\tau) - \varepsilon n\Omega\Phi(I_1)\cos(m\vartheta - n\tau), \end{aligned} \tag{5.6}$$

where Φ' denotes derivative with respect to I_1 and we have used (5.5) to replace $\langle H\rangle_{\phi_f}$. If we introduce the first of (5.6) in the Hamiltonian $\tilde{H}$ and expand $H_0(I_1 + \varepsilon \dots)$ and $\varepsilon V_1(I_1 + \varepsilon \dots)$ around I_1 up to $\mathcal{O}(\varepsilon)$ we get[1]

$$\begin{aligned} \tilde{H} = H_0(I_1) + \overbrace{\left(\frac{\partial H_0}{\partial I}\right)_{I_1}}^{\omega_0(I_1)} \varepsilon m\Phi(I_1)\cos(m\vartheta - n\tau) + \\ \varepsilon V_1(I_1)\cos(m\vartheta - n\tau) - \varepsilon n\Omega\Phi(I_1)\cos(m\vartheta - n\tau) + \mathcal{O}(\varepsilon^2) \quad . \end{aligned} \tag{5.7}$$

Thus terms of $\mathcal{O}(\varepsilon)$ in (5.7) can be killed if we choose $\Phi(I_1)$ such that

$$m\omega_0(I_1)\Phi(I_1) + V_1(I_1) - n\Omega\Phi(I_1) = 0, \quad \to \quad \Phi(I_1) = -\frac{V_1(I_1)}{m\omega_0(I_1) - n\Omega},$$

but we start assuming $|m\omega_0(I) - n\Omega| = |m\omega_0(I_1) - n\Omega| + \mathcal{O}(\varepsilon) \ll 1$!!

This *small denominator*, due to resonances prevents the above technique and averaging. In the general case (5.1), maybe in the first canonical transformation we could avoid any resonance, but since resonance are dense in phase space, in some of the sequence of canonical transformations the system could fall in some resonance, the small denominators would appear and therefore the series discussed in the previous chapters would become divergent.

Let us show how this happens using the real formulation of (5.1) and take $-n$ instead of n just to emphasize the resonant character,

$$H(I, \vartheta, \tau) = H_0(I) + \varepsilon \sum_{m,n\neq(0,0)} V_{mn}(I)\cos(m\vartheta - n\tau). \tag{5.8}$$

[1] In $V_1(I)$ we just replace I by I_1 since the correction is of $\mathcal{O}(\varepsilon^2)$.

Perform a canonical transformation $(I, \vartheta) \to (I_1, \vartheta_1)$ by introducing the generating function

$$F(I_1, \vartheta, \tau) = I_1\vartheta + \varepsilon\Theta(I_1, \vartheta, \tau),$$

where, as before,

$$I = I_1 + \varepsilon\Theta_\vartheta, \quad \tilde{H} = H + \varepsilon\Omega\Theta_\tau,$$

$\Theta(I_1, \vartheta, \tau)$ will be chosen in such a way so as to kill the terms of $\mathcal{O}(\varepsilon)$, but we will keep also terms up to $\mathcal{O}(\varepsilon^2)$. The subscripts in Θ denotes derivative with respect to the corresponding variable. Let us work first with H_0,

$$H_0(I = I_1 + \varepsilon\Theta_\vartheta) = H_0(I_1) + \varepsilon\omega_0(I_1)\Theta_\vartheta + \frac{\varepsilon^2}{2}\omega_0'(I_1)(\Theta_\vartheta)^2 + \mathcal{O}(\varepsilon^3),$$

where $\omega_0'(I_1) = (\mathrm{d}\omega_0/\mathrm{d}I)_{I_1}$. Now, let us take $\varepsilon V_{mn}(I)$,

$$\varepsilon V_{mn}(I) = \varepsilon V_{mn}(I_1) + \varepsilon^2 V_{mn}'(I_1)\Theta_\vartheta + \mathcal{O}(\varepsilon^3).$$

Finally, $\tilde{H}$ up to $\mathcal{O}(\varepsilon^2)$ reads

$$\begin{aligned}\tilde{H} &= H_0(I_1) + \varepsilon\left(\omega_0(I_1)\Theta_\vartheta + \sum_{m,n\neq(0,0)} V_{mn}(I_1)\cos(m\vartheta - n\tau) + \Omega\Theta_\tau\right) + \\ &+ \varepsilon^2\left(\frac{1}{2}\omega_0'(I_1)(\Theta_\vartheta)^2 + \sum_{m,n\neq(0,0)} V_{mn}'(I_1)\Theta_\vartheta\cos(m\vartheta - n\tau)\right) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

In order to kill the terms of $\mathcal{O}(\varepsilon)$ we set

$$\omega_0(I_1)\Theta_\vartheta + \sum_{m,n\neq(0,0)} V_{mn}(I_1)\cos(m\vartheta - n\tau) + \Omega\Theta_\tau = 0.$$

Take

$$\Theta(I_1, \vartheta, \tau) = \sum_{m,n\neq(0,0)} B_{mn}(I_1)\sin(m\vartheta - n\tau),$$

and just look for the coefficients $B_{mn}(I_1)$. Indeed, since

$$\Theta_\vartheta = \sum_{m,n\neq(0,0)} mB_{mn}(I_1)\cos(m\vartheta - n\tau),$$

$$\Theta_\tau = -\sum_{m,n\neq(0,0)} nB_{mn}(I_1)\cos(m\vartheta - n\tau),$$

then

$$B_{mn}(I_1) = -\frac{V_{mn}(I_1)}{m\omega_0(I_1) - n\Omega},$$

and finally

$$\Theta(I_1, \vartheta, \tau) = -\sum_{m,n\neq(0,0)} \frac{V_{mn}(I_1)}{m\omega_0(I_1) - n\Omega} \sin(m\vartheta - n\tau).$$

The above expression is only valid if $|m\omega_0(I_1) - n\Omega| > d > 0$[2], this means far from the *first order* resonances, that are dense in $I \in G \subset \mathbb{R}$. Moreover, at $\mathcal{O}(\varepsilon^2)$, terms like

$$(\Theta_\vartheta)^2 = \sum_{m,n,m',n'\neq 0} mB_{mn}(I_1)m'B_{m'n'}(I_1)\cos(m\vartheta - n\tau)\cos(m'\vartheta - n'\tau)$$

appear, which after a simple trigonometric manipulation can be written as

$$\frac{1}{2}\sum_{m,n,m',n'\neq 0} mB_{mn}(I_1)m'B_{m'n'}(I_1)\cos((m \pm m')\vartheta - (n \pm n')\tau).$$

Since $m \pm m', n \pm n' \in \mathbb{Z}$ we conclude that at $\mathcal{O}(\varepsilon)$ as well as at $\mathcal{O}(\varepsilon^2)$, the first order and *second order* resonances, as well as at any order in ε, they are just linear combinations of the form

$$p\,\omega_0(I_1) - q\,\Omega = 0, \qquad p, q \in \mathbb{Z}, \tag{5.9}$$

that are everywhere dense in any open set $G \in \mathbb{R}$. Even if we perform the transformation $\vartheta \to \vartheta_1$, it is clear that this only applies to the terms of $\mathcal{O}(\varepsilon)$ in the Hamiltonian $\tilde{H}$, to the term $\cos(m\vartheta - n\tau)$, that following the procedure used when we discussed perturbations to integrable systems, leads to $\cos(m\vartheta - n\tau) \approx \cos(m\vartheta_1 - n\tau) - \sin(m\vartheta_1 - n\tau)(\vartheta - \vartheta_1) \approx \cos(m\vartheta_1 - n\tau) + \mathcal{O}(\varepsilon)$. In the terms of $\mathcal{O}(\varepsilon^2)$ we just replace ϑ by ϑ_1. So this extra term does not provide any new harmonic or resonance in $\tilde{H}$ besides that defined in (5.9).

For instance, let us consider as unperturbed Hamiltonian a quartic oscillator, $H_0(I) = AI^{4/3}$, then, $\omega_0(I) = (4/3)AI^{1/3}$ and therefore, all the above formulation is not applicable when, accordingly to (5.9), $I \in (I_* - \delta, I_* + \delta), \delta > 0$

[2]Later we will briefly discuss this condition in the framework of the so-called KAM theory.

with $I_* \in \mathcal{R}$, where

$$\mathcal{R} = \left\{ I_{pq} \in \mathbb{R} : I_{pq} = \left(\frac{3q\Omega}{4Ap} \right)^3 ,\ p, q \in \mathbb{Z}/\{0\} \right\}.$$

It becomes clear then that to find initial condition "far away" from a resonance is not an easy task.

5.2 General description of a non-linear resonance

We have already seen that we cannot apply the perturbation techniques described above and averaging when the system is close to a resonance. Therefore, since in this case it is not possible to kill the terms of $\mathcal{O}(\varepsilon)$ in the perturbation, let us seek for a different, local approach. We star with the original Hamiltonian

$$H(I, \vartheta, \tau) = H_0(I) + \varepsilon \sum_{m,n \neq (0,0)} V_{mn}(I) e^{i(m\vartheta + n\tau)}, \tag{5.10}$$

and assume we are restricted to a small domain $D = (I_r - \delta, I_r + \delta) \subset \mathbb{R}$ with $|I_r| > c > 0$ some resonant action and δ "small", and for any $I \in D$ only a single pair m, n leads to $|m\dot{\vartheta} + n\dot{\tau}| \ll 1$. Again we neglect harmonics of the form $|km\dot{\vartheta} + kn\dot{\tau}|$, $k \in \mathbb{Z}$, $k > 1$. As we did before, let us take the real Fourier expansion of (5.10) and change $n \to -n$. Therefore, since only one term corresponds to a slow phase, we can average over all fast phases and we get[3]

$$H(I, \vartheta, \tau) = H_0(I) + \varepsilon V_{mn}(I) \cos(m\vartheta - n\tau). \tag{5.11}$$

Let us denote with $\psi = m\vartheta - n\tau$ the slow phase. Since we assume that for $I \in D, |\dot{\psi}| = |m\dot{\vartheta} + n\dot{\tau}| \ll 1$, let us call ψ the *resonant phase*. Now, perform a local canonical transformation, $(I, \vartheta) \to (p, \psi)$ through a generating function depending on the old action and the new phase,

$$F(I, \psi, \tau) = -(I - I_r) \underbrace{\frac{\psi + n\tau}{m}}_{\vartheta}.$$

[3]By an abuse of notation we denote the average Hamiltonian as $H(I, \vartheta, \tau)$ instead of $\langle H \rangle_{\phi_f}$ where ϕ_f denotes all fast phases.

Therefore, the transformation equations are

$$\begin{aligned} p &= -\frac{\partial F}{\partial \psi}, \quad \rightarrow \quad p = \frac{I - I_r}{m}, \quad \rightarrow \quad I = I_r + mp, \quad |mp| < \delta; \\ \vartheta &= -\frac{\partial F}{\partial I} \quad \rightarrow \quad \vartheta = \frac{\psi + n\tau}{m}, \quad \rightarrow \quad \psi = m\vartheta - n\tau; \\ H_r(p,\psi) &= H(I(p), \vartheta(\psi), \tau(\psi)) + \Omega \frac{\partial F}{\partial \tau}, \end{aligned} \tag{5.12}$$

where the subscript r stands for the "resonant Hamiltonian" that should describe the motion in D. Thus,

$$H_r(p,\psi) = H_0(I_r + mp) + \varepsilon V_{mn}(I_r + mp)\cos\psi - np\Omega. \tag{5.13}$$

Now we Taylor expand $H_0(I_r + mp)$ and $V_{mn}(I_r + mp)$ up to the minimum possible order in mp. This minimum order is $(mp)^2$ as we shall see. Since $\mathcal{O}(m^2p^2) \sim \mathcal{O}(\delta^2)$ we take then $\mathcal{O}(\delta) \sim \mathcal{O}(\sqrt{\varepsilon})$. Thus all this formulation is valid within a domain of size $\sqrt{\varepsilon}$ around I_r. Let us then expand (5.13),

$$H_r(p,\psi) = H_0(I_r) + \left(\frac{\partial H_0}{\partial I}\right)_{I_r} mp + \frac{1}{2}\left(\frac{\partial^2 H_0}{\partial I^2}\right)_{I_r} m^2p^2 + \varepsilon V_{mn}(I_r)\cos\psi - np\Omega,$$

where in V_{mn} we just take the zero order in the expansion since $\varepsilon\, mp \sim \mathcal{O}(\varepsilon^{3/2})$. Neglecting constant terms and recalling that $\partial H_0/\partial I = \omega_0$ we get

$$H_r(p,\psi) = (m\omega_0(I_r) - n\Omega)\, p + \frac{1}{2}\left(\frac{\partial \omega_0}{\partial I}\right)_{I_r} m^2p^2 + \varepsilon V_{mn}(I_r)\cos\psi; \tag{5.14}$$

since I_r is the exact resonant value, the linear term in p vanishes and denoting the constant factor $(\partial\omega_0/\partial I)_{I_r}\, m^2 = M^{-1}$ we obtain the final expression for the resonant Hamiltonian,

$$H_r(p,\psi) = \frac{p^2}{2M} + \varepsilon V_{mn}(I_r)\cos\psi. \tag{5.15}$$

Therefore the resonant Hamiltonian that describe the resonant dynamics in D is a pendulum Hamiltonian. Notice that the original Hamiltonian (5.10) depends on time and therefore, in general, non-integrable. The averaged Hamiltonian (5.11) is also time-dependent, but it is clear that it is integrable, since it reduces to the pendulum Hamiltonian (5.15) after a suitable canonical

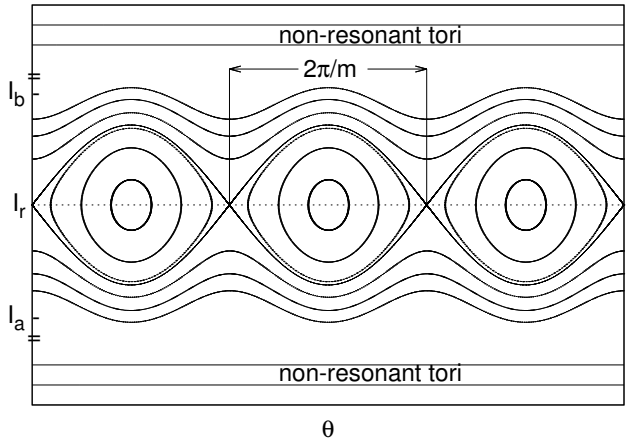

Figure 5.2: Illustration of the phase space structure near a non-linear resonance. In this example, $m = 3$. In $D = (I_a, I_b) \sim \mathcal{O}(\sqrt{\varepsilon})$ the system is *trapped* in the resonance, $\vartheta(t) \approx \omega_0(I_r)t + \psi(t)/m$, the topology changes, while if $I \notin D$, $\vartheta(t) \approx \omega_0(I)t$ and the tori structure of H_0 is preserved.

transformation. Notice that the "mass" M thus defined, measures the non–linear character of the oscillations (or rotations). Indeed, $M^{-1} \propto (\partial\omega_0/\partial I)$ and thus it becomes clear that (5.15) is not applicable when H_0 is nearly linear. In the new variables (p, ψ), we know completely the solution of (5.15), each trajectory is labeled by the local integral $H_r(p, \psi) = h_*$. Let us assume that $M > 0$, $\varepsilon V_{mn}(I_r) > 0$, then for $h_* < \varepsilon V_{mn}(I_r)$ the system oscillates around $p = 0$ $(I_r), \psi = \pi$, while for $h_* > \varepsilon V_{mn}(I_r)$ it rotates. Moreover, for h_* small, we have the small oscillation regime with a small oscillation frequency given by

$$\Omega_\psi^2(\varepsilon) = \varepsilon \frac{V_{mn}(I_r)}{M} = \varepsilon V_{mn}(I_r) m^2 \left(\frac{\partial \omega_0}{\partial I}\right)_{I_r},$$

after replacing M^{-1}. Therefore, we get for the small oscillation frequency of the phase ψ

$$\Omega_\psi(\varepsilon) = m\sqrt{\varepsilon V_{mn}(I_r)\omega_0'(I_r)}, \tag{5.16}$$

where the prime denotes derivative with respect to I. Fig. 5.2 illustrates how the resonant dynamics looks like. If $\varepsilon = 0$, the tori structure, as represented in this figure, corresponds to horizontal lines and $\vartheta(t) = \omega_0(I)t + \vartheta_0$. For $\varepsilon \neq$

0, the tori structure is preserved for I away from D (by means of averaging we can eliminate the perturbation and the average Hamiltonian is H_0), but for $I \in D = (I_a, I_b) \sim \mathcal{O}(\sqrt{\varepsilon})$, the perturbation cannot be killed due to the presence of a resonance and the topology of the phase space changes, $\vartheta(t) = n\tau/m + \psi/m \approx \omega_0(I_r)t + \psi(t)/m$, where we have replaced $n\Omega/m$ by $\omega_0(I_r)$ due to the resonance condition. Therefore the motion is a composition of a rotation and a pendulum. Thus, near the resonance, the frequency, $\dot{\vartheta}$ reads

$$\omega_0 \approx \omega_{0r} + \frac{\dot{\psi}}{m} = \omega_{0r} + \frac{p}{mM},$$

where we have used (5.15) to replace $\dot{\psi}$ using the Hamilton equations, and recalling the definition of M we finally get

$$\omega_0(I,t) \approx \omega_0(I_r) + m\omega_0'(I_r)p(t). \tag{5.17}$$

Clearly $\omega_0(I,t)$ is not constant, it changes with time with the pendulum momentum p. The same happens to the unperturbed action I, accordingly to the first in (5.12), $I = I_r + mp$.

5.2.1 Resonance half-width

As we have already seen, the change of topology in phase space near a resonance occurs within the oscillation regime of the pendulum, as it is observed in Fig. 5.2. Therefore, from the resonant Hamiltonian (5.15), the energy label $h_* = \varepsilon V_{mn}(I_r)$ corresponds to the separatrix, then from

$$\frac{p_s^2}{2M} + \varepsilon V_{mn}(I_r)\cos\psi_s = \varepsilon V_{mn}(I_r), \quad \rightarrow \quad \frac{p_s^2}{2M} = 2\varepsilon V_{mn}(I_r)\sin^2\frac{\psi_s}{2},$$

after a simple trigonometric relation. Therefore the separatrix equation reads

$$p_s = \pm p_r \sin\frac{\psi_s}{2}, \qquad p_r = 2\sqrt{\varepsilon V_{mn}(I_r)M} \sim \mathcal{O}(\sqrt{\varepsilon}). \tag{5.18}$$

In the original variables (I, ϑ, τ), we have $I = I_r + mp$, thus $I_s = I_r + mp_s$ therefore

$$|I - I_r|_{\max} \equiv (\Delta I)_r = mp_r \sim \mathcal{O}(\sqrt{\varepsilon}). \tag{5.19}$$

$(\Delta I)_r$ is called the half-width of the resonance in action space. As a difference from the non-resonant case, where the change in the unperturbed action is of

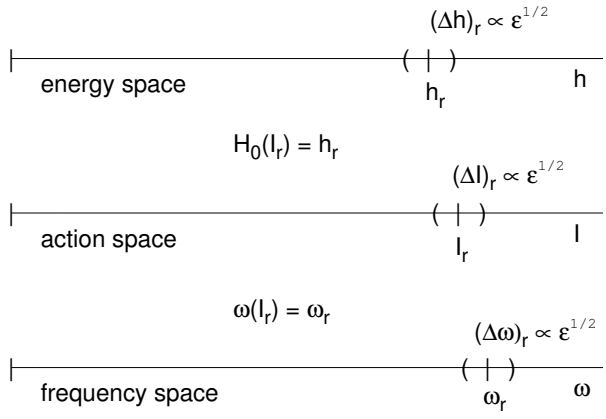

Figure 5.3: Sketch of the half-width of a non-linear resonance in action, frequency and energy spaces.

$\mathcal{O}(\varepsilon)$, the effect of a resonant perturbation is $\mathcal{O}(\sqrt{\varepsilon})$, larger than the former since $\varepsilon \ll 1$.

In the very same way we can define the half-width in frequency space. Indeed, from (5.17) we set $p = p_s$ and thus

$$|\omega(I_s) - \omega(I_r)|_{\max} \equiv (\Delta\omega)_r = m\omega_0'(I_r)p_r = 2\sqrt{\frac{\varepsilon V_{mn}(I_r)}{M}} \sim \mathcal{O}(\sqrt{\varepsilon}) \quad (5.20)$$

where $(\Delta\omega)_r$ is the half-width of the resonance in frequency space. In a similar fashion we can define the half-width of the resonance in energy space from $H_0(I_r + mp_s)$ as $(\Delta h)_r = m\omega_0(I_r)p_r \sim \mathcal{O}(\sqrt{\varepsilon})$, after linearizing H_0 around I_r and $H_0(I_r) = h_r$.

To end this section, let us focus on the resonant Hamiltonian (5.15). Since we have assumed that $M > 0$ and $\varepsilon V_{mn}(I_r) > 0$, from the pendulum dynamics, it is clear that for $h_* = -\varepsilon V_{mn}(I_r)$ $p = 0, \psi = \pi$ is the stable equilibrium point, while for $h_* = \varepsilon V_{mn}(I_r)$ $p = 0, \psi = 0$ corresponds to the unstable equilibrium point or whiskered torus. The stable point, that corresponds to the minimum h_*, leads to the elliptic resonant tori of dimension 1

$$p = 0 \quad \rightarrow \quad I = I_r, \qquad \psi = \pi \quad \rightarrow \vartheta = \frac{\pi}{m} + \frac{n}{m}\tau, \quad \vartheta, \tau \bmod 2\pi;$$

while the whiskered (or hyperbolic) torus, that appears for the maximum h_*

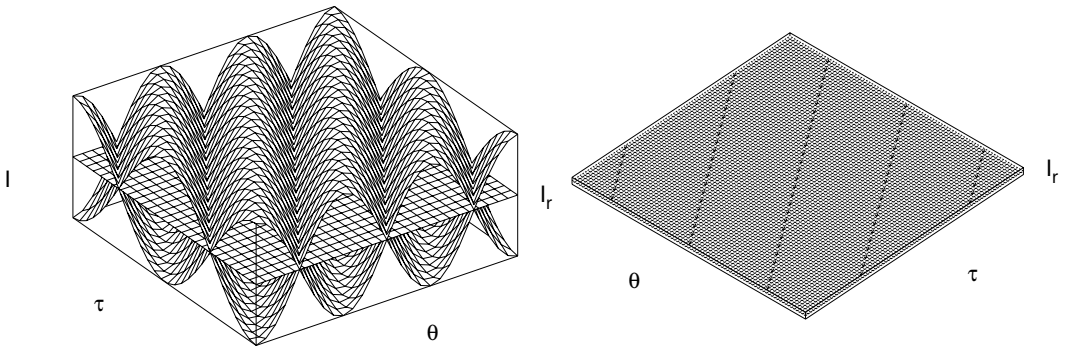

Figure 5.4: Whiskers and the whiskered torus for $m = 3, n = 2$ for a 3-D phase space. The whiskers $I^{u,s}$ are shown at the left. They intersect leading to the whiskered torus plotted at the right.

within the oscillation regime, in the original variables is,

$$p = 0, \quad \rightarrow \quad I = I_r,$$

$$\psi = 0, \quad \rightarrow \quad m\vartheta - n\tau = 0, \quad \rightarrow \quad \vartheta = \frac{n}{m}\tau, \quad \vartheta, \tau \bmod 2\pi;$$

which is a whiskered torus of dimension 1 and the whiskers, of dimension 2,

$$I^{u,s} = I_r \pm m p_r \sin\left(\frac{m\vartheta - n\tau}{2}\right)$$

are fastened to it.

To illustrate this, in Fig. (5.4) we plot at the left the two whiskers $I^{u,s} = I_r \pm m p_r \sin(m\vartheta/2 - n\tau/2)$ for $m = 3, n = 2$. The arriving whisker or stable manifold I^s and the unstable manifold or departing whisker I^u intersect each other at I_r leading to the whiskered torus, which is represented in Fig. (5.4) at the right. The motion in the whiskered (resonant) tori, $I = I_r$ is given by $\vartheta = n\tau/m$, and therefore the orbit close itself, it is not dense on the torus. Clearly the same happens with the motion on the elliptic resonant tori, $\vartheta = \pi/m + n\tau/m$.

5.3 Non-linear stabilization

Let us briefly discuss the differences between a linear and non-linear resonance. We have already seen that in a non-linear oscillator in resonance with an external periodic perturbation of strength ε, the unperturbed action change by an amount $\sim \sqrt{\varepsilon}$.

Now, consider the following example of a linear oscillator under the effect of a linear periodic perturbation,

$$H(I,\vartheta,\tau) = H_0(I) + \varepsilon B I \cos(m\vartheta - n\tau), \qquad H_0(I) = \omega_l I,$$

where $\tau = \Omega t + \tau_0$, ω_l the constant frequency of the linear oscillator and B a numerical constant. The resonant condition is then

$$m\omega_l - n\Omega = 0.$$

Assume that in frequency space $\omega_l \approx n\Omega/m$, then $m\vartheta - n\tau \equiv \psi$ is the resonant phase. Following the same approach than for the non-linear resonance, let us introduce a canonical transformation through the generating function

$$F(I,\psi,\tau) = -I\frac{\psi + n\tau}{m},$$

where now, since the resonance condition is global in action space, there is no resonant action I_r. Therefore the transformation equations are

$$\begin{aligned}
p &= -\frac{\partial F}{\partial \psi}, \quad \rightarrow \quad p = \frac{I}{m}, \quad \rightarrow \quad I = mp, \\
\vartheta &= -\frac{\partial F}{\partial I} \quad \rightarrow \quad \vartheta = \frac{\psi + n\tau}{m}, \quad \rightarrow \quad \psi = m\vartheta - n\tau; \\
{H^l}_r(p,\psi) &= H(I(p),\vartheta(\psi),\tau(\psi)) + \Omega\frac{\partial F}{\partial \tau},
\end{aligned}$$

where ${H^l}_r$ denotes the linear resonant Hamiltonian. Since both $H_0(I)$ and the perturbation are linear in I there is no need to perform any Taylor expansion and we get

$${H^l}_r(p,\psi) = (m\omega_l - n\Omega)p + \varepsilon m B p \cos\psi.$$

Again we get an integrable Hamiltonian, each solution depends on the energy levels $H^l_r(p,\psi) = h_*$. However, while in the non-linear case H_r is a local

integral around I_r (or h_r), H_r^l is global for any I (or p). Since we assumed that $|m\omega_l - n\Omega| = \Delta \ll 1$, for any energy level we have

$$p(\Delta + \varepsilon m B \cos\psi) = h_*.$$

For ω_l close enough to $n\Omega/m$, the condition $\Delta < \varepsilon m B$ holds, and then, for any finite value of h_*, the amplitude of the oscillations, p, grows unboundedly when the term between brackets goes to zero. Therefore the linear oscillations become highly unstable.

The main difference between a non-linear and linear resonance is the fact that for a non-linear oscillator the frequency depends on the action or the energy. Therefore a periodic time dependent perturbation slightly changes the energy of the system, and as (5.17) shows, it does the frequency. The perturbation periodically moves the system away from the exact resonance value and this leads to a kind of non-linear stabilization in contrast to what happens in the linear case.

Physically speaking, a resonance means an exchange of energy between the oscillator and the perturbation. When the oscillator and the external perturbation are in resonance, the oscillator takes energy from the external periodic force. Therefore if the oscillator is linear and in resonance, it will gain energy in each period of the perturbation, while in the case of a non-linear one, the system is periodically driven out and in the resonance and therefore the energy exchange is finite.

5.4 Ranges of application of the pendulum model

To close this chapter, let us discuss some restrictions to the pendulum model to model a non-linear resonance. To this aim let us use the non-linear parameter introduced in a previous chapter,

$$\alpha(I) = \frac{I}{\omega(I)}\omega'(I),$$

where prime denotes derivative with respect to I. Thus, we will rewrite relevant parameters derived above in terms of α.

Recall the small oscillation frequency defined in (5.16)

$$\Omega_\psi(\varepsilon) = m\sqrt{\varepsilon V_{mn}(I_r)\omega_0'(I_r)}.$$

Rescale ε such that

$$\epsilon = \varepsilon \frac{V_{mn}(I_r)}{I_r \omega_0(I_r)},$$

which is possible since all the involved quantities are finite, so $\mathcal{O}(\epsilon) = \mathcal{O}(\varepsilon)$. Therefore

$$\Omega_\psi(\epsilon) = m\sqrt{\epsilon I_r \omega_0(I_r)\omega_0'(I_r)}, \quad \rightarrow \quad \Omega_\psi(\epsilon) = m\omega_0(I_r)\sqrt{\epsilon\alpha(I_r)}.$$

On the other hand, the half-width of the resonance in action space given by (5.19)

$$|I - I_r|_{\max} = mp_r, \quad p_r = 2\sqrt{\varepsilon V_{mn}(I_r) M},$$

introducing the rescaled perturbation parameter ϵ, using α and the definition of M it is straightforward to get

$$p_r = I_r \frac{2}{m}\sqrt{\frac{\epsilon}{\alpha(I_r)}}.$$

i) The pendulum approximation rests on the smallness of mp in such a way that we can neglect terms higher than p^2. Therefore it is required that

$$mp \leq mp_r \ll 1 \quad \rightarrow \quad 2I_r\sqrt{\frac{\epsilon}{\alpha(I_r)}} \ll 1 \quad \rightarrow \quad \epsilon \ll \alpha(I_r).$$

Thus, nonlinearity should be not too small.
ii) The frequency of the resonant phase should be slow enough such that we could average out the rest of the phases we assumed as fast. Since the oscillation frequency of the pendulum is bounded from above by Ω_ψ, we require that

$$\Omega_\psi(\epsilon) \ll m'\omega(I) - n'\Omega \equiv \omega_{m'n'}, \qquad m',\, n' \in \mathbb{Z},$$

thus,

$$m\omega_0(I_r)\sqrt{\epsilon\alpha(I_r)} \ll \omega_{m'n'} \quad \rightarrow \quad \alpha(I_r) \ll \frac{1}{\epsilon}\left(\frac{\omega_{m'n'}}{m\omega_0(I_r)}\right)^2 \sim \frac{1}{\epsilon}.$$

Therefore, the non-linearity should be not too large.

Thus we can say that the pendulum model is a good formulation of a non-linear resonance in the case of moderate nonlinearity.

Chapter 6

Multidimensional non-linear resonance

Let us consider the N-D Hamiltonian $\mathcal{H}_0(\boldsymbol{P}, \boldsymbol{X})$ where $\boldsymbol{P}, \boldsymbol{X} \in F \subset \mathbb{R}^N$. Assume we know the canonical transformation to action-angle variables $(\boldsymbol{P}, \boldsymbol{X}) \to (\boldsymbol{I}, \boldsymbol{\vartheta})$ such that $\mathcal{H}_0(\boldsymbol{P}, \boldsymbol{X}) \to H_0(\boldsymbol{I})$. Let now be

$$\mathcal{H}_0(\boldsymbol{P}, \boldsymbol{X}) + \varepsilon \mathcal{V}(\boldsymbol{X}),$$

where $\varepsilon \ll 1$ and $\mathcal{V}(\boldsymbol{X})$ is a well behaved function of the coordinates. Then, since we know the transformation $(\boldsymbol{P}, \boldsymbol{X}) \to (\boldsymbol{I}, \boldsymbol{\vartheta})$, the *perturbation* can be written in terms of the action-angle variables, $\mathcal{V}(\boldsymbol{X}) \to V(\boldsymbol{I}, \boldsymbol{\vartheta})$. The latter admits a Fourier expansion in the angles and therefore the Hamiltonian takes the form

$$H(\boldsymbol{I}, \boldsymbol{\vartheta}) = H_0(\boldsymbol{I}) + \varepsilon \sum_{\boldsymbol{m} \neq 0} V_{\boldsymbol{m}}(\boldsymbol{I}) e^{i\boldsymbol{m}\cdot\boldsymbol{\vartheta}}, \qquad \varepsilon \ll 1, \tag{6.1}$$

$\boldsymbol{I}$ defined in the N-dimensional manifold $G = \{\boldsymbol{I} = (I_1, \ldots, I_N)\} \subset \mathbb{R}^N$, $\boldsymbol{\vartheta}$ in the N-dimensional torus $S^N = \{\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_N) \bmod 2\pi\}$. Besides, $\boldsymbol{m} \in \mathbb{Z}^N/\{0\}$ and $V_{\boldsymbol{m}}$ are complex coefficients that depend on the action and we have assumed that the mean value $V_{\boldsymbol{0}} = 0$ without any loss of generality.

6.1 Integrable systems

Let us briefly summarize the dynamics when $\varepsilon = 0$. Thus $H = H_0(\boldsymbol{I})$ and the system is completely integrable with N global integrals (in involution)

$I_1, I_2, \ldots, I_N$. The motion is in general *quasiperiodic* with N fundamental frequencies

$$\boldsymbol{\omega} = \frac{\partial H_0}{\partial \boldsymbol{I}} \in G^*$$

or

$$(\omega_1, \omega_2, \ldots, \omega_N) = \left(\frac{\partial H_0}{\partial I_1}, \frac{\partial H_0}{\partial I_2}, \ldots, \frac{\partial H_0}{\partial I_N} \right).$$

In this case of "independent" N frequencies, the motion fills densely and uniformly an N-dimensional torus. We assume that the determinant

$$\det \left(\frac{\partial \omega_i}{\partial I_j} \right) = \det \left(\frac{\partial^2 H_0}{\partial I_j \partial I_i} \right) \neq 0, \qquad i, j = 1, \ldots, N,$$

in order $\boldsymbol{\omega}(\boldsymbol{I})$ to be a one-to-one application and therefore invertible. This condition, in fact, determines the nonlinearity of the oscillations, for instance $N = 1$ implies $\mathrm{d}\omega/\mathrm{d}I \neq 0$ which is the nonlinear condition assumed in the previous chapter. Therefore, as we did before, we can look the dynamics either on the action or frequency spaces. Thus an N-torus is completely specified by the initial conditions that fix the set of the N values of the actions I_i or the frequencies ω_i. In other words, the coordinates that define a N-torus are the actions or the frequencies. The phase space $G \times S^N$ is completely foliated by N-dimensional invariant tori.

A *resonance condition* in H_0 takes the form

$$\boldsymbol{m} \cdot \boldsymbol{\omega}(\boldsymbol{I}) = m_1 \omega_1(\boldsymbol{I}) + m_2 \omega_2(\boldsymbol{I}) + \cdots + m_N \omega_N(\boldsymbol{I}) = 0, \tag{6.2}$$

in action space, while

$$\boldsymbol{m} \cdot \boldsymbol{\omega} = m_1 \omega_1 + m_2 \omega_2 + \cdots + m_N \omega_N = 0, \tag{6.3}$$

in frequency space. The resonance condition (6.2) or (6.3) defines the resonant torus, the latter is completely specified by the value of $\boldsymbol{I}$ or $\boldsymbol{\omega}$ that satisfies the resonance equation. Therefore, in general, on a single resonant torus, the motion is not dense on the N-torus but it is on an $(N-1)$-dimensional one. In terms of the frequency, (6.3) implies that one of the N components of the frequency vector can be expressed in terms of the rest of them, so we have $N-1$ "independent" frequencies.

This kind of *coupling resonance* in a multidimensional system takes into account the exchange of energy among the different degrees of freedom. Geometrically, in action space, the relation $H_0(\boldsymbol{I}) = h$ defines the $N-1$ dimensional manifold, called *energy surface*

$$M_h = \{\boldsymbol{I} \in G : H_0(\boldsymbol{I}) = h\}\,, \tag{6.4}$$

while the resonance condition (6.3) defines another $N-1$ dimensional manifold, called *resonant surface*

$$\Sigma_r^{\boldsymbol{m}} = \{\boldsymbol{I} \in G : \boldsymbol{m}\cdot\boldsymbol{\omega}(\boldsymbol{I}) = 0\}\,. \tag{6.5}$$

Similarly, in frequency space, since $\boldsymbol{\omega}(\boldsymbol{I})$ is one-to-one, $\boldsymbol{I}(\boldsymbol{\omega})$ does exist and we can define the $N-1$ surfaces

$$M'_h = \{\boldsymbol{\omega} \in G^* : H'_0(\boldsymbol{\omega}) = h\}\,, \qquad \Sigma'^{\boldsymbol{m}}_r = \{\boldsymbol{\omega} \in G^* : \boldsymbol{m}\cdot\boldsymbol{\omega} = 0\}\,, \tag{6.6}$$

where $H'_0(\boldsymbol{\omega}) = H_0(\boldsymbol{I}(\boldsymbol{\omega}))$. They are the energy and resonant surfaces in frequency space. The latter is just a $N-1$ dimensional plane whose normal is the *resonant vector* $\boldsymbol{m}$. Fig. 6.1 shows these two surfaces in both spaces for $N = 3$, corresponding to three uncoupled quartic oscillators. In any of these spaces the energy and resonant surfaces intersect each other, in this particular case leading to a curve, but in general this intersection is some manifold

$$\Upsilon_h^{\boldsymbol{m}} = \{\boldsymbol{I}_r : \boldsymbol{I}_r \in M_h \cap \Sigma_r^{\boldsymbol{m}}\}\,,$$

where $\dim\Upsilon_h^{\boldsymbol{m}} = N-2$. Thus all the vectors that belong to $\Upsilon_h^{\boldsymbol{m}}$ are the resonant actions $\boldsymbol{I}_r$ and those that belong to $\Upsilon'^{\boldsymbol{m}}_h \equiv M'_h \cap \Sigma'^{\boldsymbol{m}}_r$ are the resonant frequencies $\boldsymbol{\omega}_r$ for a given energy label h.

Since the frequency vector is the gradient of H_0, $\boldsymbol{\omega}(\boldsymbol{I})$ is normal to M_h at any point in action space. From the resonant condition (6.3), $\boldsymbol{m}$ is normal to $\boldsymbol{\omega}(\boldsymbol{I})$, and then tangent to the energy surface at $\boldsymbol{I}_r$. Besides, the normal to $\Sigma_r^{\boldsymbol{m}}$ is defined by

$$\boldsymbol{n}_r = \frac{\partial}{\partial \boldsymbol{I}}(\boldsymbol{m}\cdot\boldsymbol{\omega}(\boldsymbol{I})); \quad (\boldsymbol{n}_r)_i = m_j\frac{\partial\omega_j}{\partial I_i} = m_1\frac{\partial\omega_1}{\partial I_i} + m_2\frac{\partial\omega_2}{\partial I_i} + \cdots + m_N\frac{\partial\omega_N}{\partial I_i},$$

where the last expression corresponds to the i–component of $\boldsymbol{n}_r$. In general $\boldsymbol{n}_r$ at $\boldsymbol{I}_r$ is not normal to $\boldsymbol{\omega}$ and therefore $\boldsymbol{n}_r$ and $\boldsymbol{m}$ are not collinear. Fig. 6.2 represents these vectors at a given resonant action $\boldsymbol{I}_r \in \Upsilon_h^{\boldsymbol{m}}$.

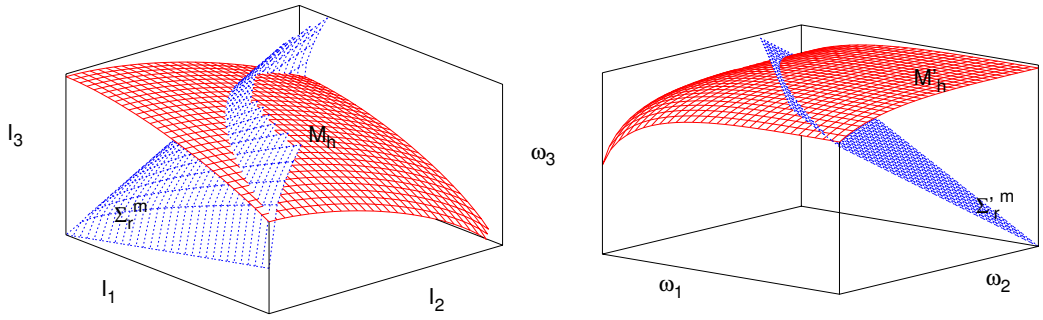

Figure 6.1: Energy and resonant surfaces in action and frequency space for a three dimensional quartic oscillator. Every point in these spaces represent a torus.

6.2 Resonant perturbation

Let us take the Hamiltonian (6.1) for $\varepsilon \neq 0$, let $D = \{\boldsymbol{I} : \|\boldsymbol{I} - \boldsymbol{I}_r\| < \delta\} \subset G$ where $\boldsymbol{I}_r \in \Upsilon_h^{\boldsymbol{m}}$ is a fixed resonant action vector and δ is "small". Assume that within D, only one term in the Fourier expansion is slow, say that involving the harmonic $\boldsymbol{m}$, so we can average out the remainder terms and considering the real part of the series, we get for $\boldsymbol{I} \in D$,

$$H(\boldsymbol{I}, \boldsymbol{\vartheta}) = H_0(\boldsymbol{I}) + \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}) \cos(\boldsymbol{m} \cdot \boldsymbol{\vartheta}). \tag{6.7}$$

The variation of the unperturbed integral $\boldsymbol{I}$ due to the presence of the perturbation to H_0 is

$$\dot{\boldsymbol{I}} = -\frac{\partial H}{\partial \boldsymbol{\vartheta}} = \varepsilon \boldsymbol{m} V_{\boldsymbol{m}}(\boldsymbol{I}) \sin(\boldsymbol{m} \cdot \boldsymbol{\vartheta}).$$

The above expression tell us that in case of a single resonant perturbation the change of the unperturbed action, $\Delta\boldsymbol{I}$, has a fixed direction along the resonant vector $\boldsymbol{m}$. Therefore the motion is confined to a one-dimensional manifold. Indeed, since at $\boldsymbol{I}_r$, $\boldsymbol{\omega}_r \perp M_h$, and $\boldsymbol{m} \perp \boldsymbol{\omega}_r$, then $\Delta\boldsymbol{I} \parallel \boldsymbol{m}$ and lies in Π_r, the tangent plane to M_h at $\boldsymbol{I}_r$. Thus, we can say that a resonant perturbation preserves the unperturbed energy.

Let us introduce a canonical transformation $(\boldsymbol{I}, \boldsymbol{\vartheta}) \to (\boldsymbol{p}, \boldsymbol{\psi})$ and instead of defining this transformation by means of a generating function, let us do

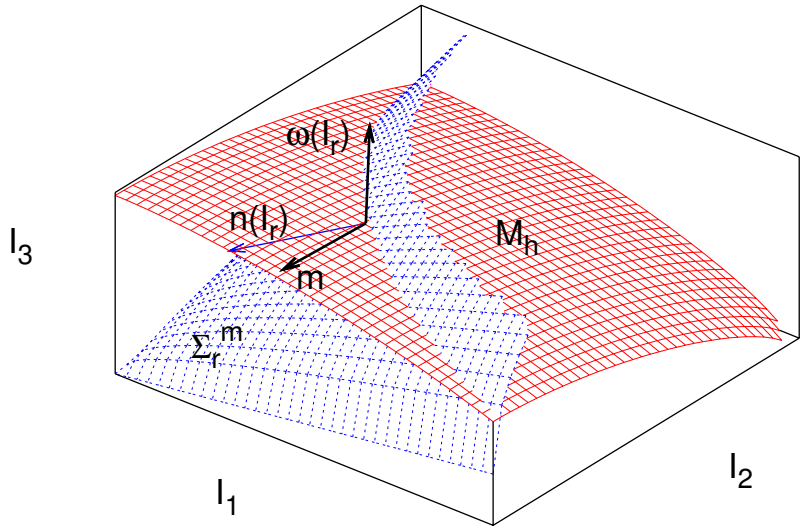

Figure 6.2: Schematic representation of the vectors $\boldsymbol{\omega}, \boldsymbol{m}$ and $\boldsymbol{n}_r$ on a given $\boldsymbol{I}_r \in \Upsilon_h^{\boldsymbol{m}}$ where $\boldsymbol{n}_r$ is not normal to $\boldsymbol{\omega}$.

it by a *local* change of basis[1]. Let

$$\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N\}$$

be the basis in which the action vector has components

$$\boldsymbol{I} = I_i \boldsymbol{u}_i, \qquad \text{sum over } i,$$

and therefore

$$\boldsymbol{I}_d \equiv \boldsymbol{I} - \boldsymbol{I}_r = (I_i - I_{ri})\boldsymbol{u}_i.$$

Let

$$\hat{\mathcal{B}} = \{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_N\}$$

be a new basis in which $\boldsymbol{I}_d$ denoted by $\boldsymbol{p}$, has components

$$\boldsymbol{I}_d \equiv \boldsymbol{p} = p_i \boldsymbol{\mu}_i.$$

[1]Later we will need the generating function that leads to the full canonical transformation.

Clearly $\boldsymbol{p}$ and $\boldsymbol{I}_d$ are the very same vector, since any vector is independent of the basis, but they have different components when they are expanded in different basis. Let us construct then the new basis $\hat{\mathcal{B}}$ accordingly to the geometry of the dynamics discussed above.

Since $\dot{\boldsymbol{I}} \parallel \boldsymbol{m}$ then $\dot{\boldsymbol{p}} \parallel \boldsymbol{m}$, thus we take $\boldsymbol{\mu}_1 = \boldsymbol{m}$. As we have already seen, $\boldsymbol{m} \perp \boldsymbol{\omega}_r$, then we take $\boldsymbol{\mu}_2 = \hat{\boldsymbol{\omega}}_r \equiv \boldsymbol{\omega}_r / \|\boldsymbol{\omega}_r\|$. The remainder vectors of the basis are defined as $\boldsymbol{\mu}_k = \boldsymbol{e}_k \in \Upsilon_h^{\boldsymbol{m}} \subset \Pi_r, k = 3, \cdots, N$, being the vectors $\boldsymbol{e}_k$ orthonormal to each other and to $\boldsymbol{\mu}_2$. One of the $\boldsymbol{e}_k$, say $\boldsymbol{e}_s$ is taken orthonormal to $\boldsymbol{n}_r$, the normal to $\Sigma_r^{\boldsymbol{m}}$. Thus, in general, all the vectors $\boldsymbol{e}_k$ will be also orthogonal to $\boldsymbol{\mu}_1$, except $\boldsymbol{e}_s$. Geometrically, for $N = 3$, we have the following picture: the resonant vector $\boldsymbol{m} \in \Pi_r$, while the frequency vector is normal to Π_r. Then the third vector, just one of the $\boldsymbol{e}_k$, $\boldsymbol{e}_s \equiv \boldsymbol{e}$ is tangent to $\Upsilon_h^{\boldsymbol{m}} \subset \Pi_r$, that is, tangent to the intersection of M_h and $\Sigma_r^{\boldsymbol{m}}$ and therefore it belongs to Π_r at $\boldsymbol{I}_r$. Fig 6.3 illustrate this local change of basis in an action space of dimension 3.

As discussed above, in general $\boldsymbol{e}_s$ and $\boldsymbol{m}$ are not orthogonal so we are dealing with a non-orthogonal basis. Therefore the N-linear-independent vectors of $\hat{\mathcal{B}}$ are

$$\hat{\mathcal{B}} = \{\boldsymbol{\mu}_1 = \boldsymbol{m},\, \boldsymbol{\mu}_2 = \hat{\boldsymbol{\omega}}_r,\, \boldsymbol{\mu}_k = \boldsymbol{e}_k, k = 3, \dots N\}\,.$$

Let us proceed with the local change of basis,

$$\text{in basis } \mathcal{B}: \qquad \boldsymbol{I}_d = I_{di}\boldsymbol{u}_i, \qquad I_{di} = I_i - I_{ri},$$

while

$$\text{in basis } \hat{\mathcal{B}}: \qquad \boldsymbol{p} = p_i \boldsymbol{\mu}_i.$$

Let us expand the vectors $\boldsymbol{\mu}_i$ in the basis $\mathcal{B}$,

$$\boldsymbol{\mu}_i = \mu_{ij}\boldsymbol{u}_j,$$

μ_{ij} are then the components of the vectors $\boldsymbol{\mu}_i$ in $\mathcal{B}$. As we said, since a vector is independent of the basis, we have

$$I_{di}\boldsymbol{u}_i = p_i\boldsymbol{\mu}_i = p_i\mu_{ij}\boldsymbol{u}_j,$$

where i, j in the third term are dummy indexes of sum and we can exchange them and write

$$I_{di}\boldsymbol{u}_i = p_j\mu_{ji}\boldsymbol{u}_i, \quad \rightarrow \quad (I_{di} - p_j\mu_{ji})\boldsymbol{u}_i = 0,$$

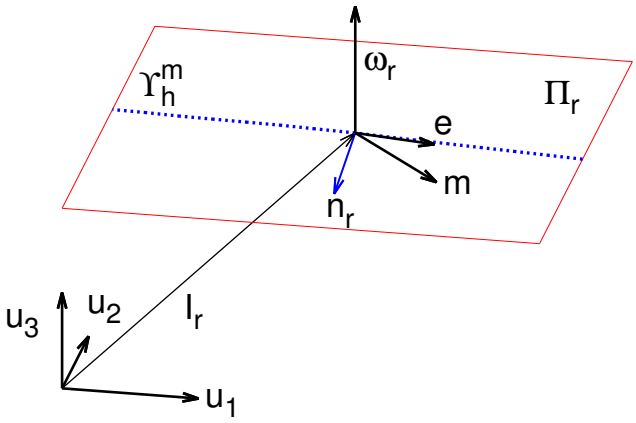

Figure 6.3: Schematic representation of the local basis $\hat{\mathcal{B}} = \{\boldsymbol{\mu}_1 = \boldsymbol{m}, \boldsymbol{\mu}_2 = \hat{\boldsymbol{\omega}}_r, \boldsymbol{\mu}_3 = \boldsymbol{e}\}$ on Π_r, the tangent plane to the energy surface M_h at $\boldsymbol{I}_r$; and the (global) basis $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. The vector $\boldsymbol{\omega}_r \perp \Pi_r$ while $\boldsymbol{m}, \boldsymbol{e} \in \Pi_r$. The vector $\boldsymbol{e}$ is defined such that $\boldsymbol{e} \perp \boldsymbol{n}_r$ ($\boldsymbol{n}_r \notin M_h$) and $\boldsymbol{e} \perp \boldsymbol{\omega}_r$, then $\boldsymbol{e}$ is tangent to $\Upsilon_h^{\boldsymbol{m}}$ ($\Upsilon_h^{\boldsymbol{m}}$ represents in the figure the projection of $\Upsilon_h^{\boldsymbol{m}} \in M_h$ on Π_r). In general, $\boldsymbol{m}$ is not orthogonal to $\boldsymbol{e}$.

and since the $\boldsymbol{u}_i$ are linear-independent, it should be

$$I_{di} = p_j \mu_{ji},$$

and then we get for the transformation of the actions

$$I_i = I_{ri} + p_j \mu_{ji}.$$

Since we impose that $(\boldsymbol{I}, \boldsymbol{\vartheta}) \to (\boldsymbol{p}, \boldsymbol{\psi})$ is a canonical transformation, it should exist a generation function, for instance $F(\boldsymbol{p}, \boldsymbol{\vartheta})$ such that

$$I_l = \frac{\partial F}{\partial \vartheta_l}, \qquad \psi_k = \frac{\partial F}{\partial p_k}.$$

Thus if we replace in the first of above

$$I_{rl} + p_j \mu_{jl} = \frac{\partial F}{\partial \vartheta_l}$$

and we integrate respect to ϑ_l we get

$$F(\boldsymbol{p}, \boldsymbol{\vartheta}) = (I_{rl} + p_j \mu_{jl})\vartheta_l + G(\boldsymbol{p}, \vartheta_1, \vartheta_2 \ldots, \vartheta_{l-1}, \vartheta_{l+1}, \ldots, \vartheta_N),$$

where G needs to be determined and for the time being, no sum over the repeated index l should be considered. But, for any other component of the action I_s, $s \neq l$ it is

$$I_{rs} + p_j \mu_{js} = \frac{\partial F}{\partial \vartheta_s} = \frac{\partial G}{\partial \vartheta_s}.$$

Therefore, since $\partial \vartheta_l / \partial \vartheta_s = \delta_{ls}$, the generating function F is then

$$F(\boldsymbol{p}, \boldsymbol{\vartheta}) = (I_{rl} + p_j \mu_{jl})\vartheta_l + G_1(\boldsymbol{p}),$$

where now the sum over both repeated indexes j, l should be considered and the new function G_1 might depends only on $\boldsymbol{p}$. The latter is fixed by the condition

$$\frac{\partial F}{\partial p_k} \equiv \psi_k = \mu_{jl}\vartheta_l \delta_{jk} + \frac{\partial G_1}{\partial p_k},$$

and thus we take F as simple as possible by setting $G_1 = \text{cte}$ and then

$$\psi_k = \mu_{kl}\vartheta_l.$$

Therefore, the complete set of transformations are

$$I_i = I_{ri} + p_j \mu_{ji}, \qquad \psi_k = \mu_{ki}\vartheta_i, \quad i, j, k = 1, 2, \ldots, N; \tag{6.8}$$

where

$$\mu_{1i} = m_i, \qquad \mu_{2i} = \hat{\omega}_{ri}, \qquad \mu_{ki} = e_{ki}, \quad k > 2. \tag{6.9}$$

In terms of all components, for $N = 3$, the explicit transformation is

$$\begin{aligned} I_1 &= I_{r1} + p_1\mu_{11} + p_2\mu_{21} + p_3\mu_{31}, \\ I_2 &= I_{r2} + p_1\mu_{11} + p_2\mu_{22} + p_3\mu_{32}, \\ I_3 &= I_{r3} + p_1\mu_{13} + p_2\mu_{23} + p_3\mu_{33}, \end{aligned}$$

$$\begin{aligned} \psi_1 &= \mu_{11}\vartheta_1 + \mu_{12}\vartheta_2 + \mu_{13}\vartheta_3, \\ \psi_2 &= \mu_{21}\vartheta_1 + \mu_{22}\vartheta_2 + \mu_{23}\vartheta_3, \\ \psi_3 &= \mu_{31}\vartheta_1 + \mu_{32}\vartheta_2 + \mu_{33}\vartheta_3. \end{aligned}$$

Thus, it is clear that the phases transform through the matrix $\mathbb{M} = \{\mu_{ij}\}$, such that $\boldsymbol{\psi} = \mathbb{M}\boldsymbol{\vartheta}$ and the actions do so as $\boldsymbol{I} = \boldsymbol{I}_r + \mathbb{M}^T\boldsymbol{p}$. From (6.8) and (6.9) it turns out that

$$\psi_1 = \mu_{1i}\vartheta_i = m_i\vartheta_i = \boldsymbol{m}\cdot\boldsymbol{\vartheta},$$

and therefore the Hamiltonian (6.7) depends on only one phase, ψ_1. Indeed, if we perform the transformation $(\boldsymbol{I}, \boldsymbol{\vartheta}) \to (\boldsymbol{p}, \boldsymbol{\psi})$ in (6.7) we have

$$H_r(\boldsymbol{p}, \boldsymbol{\psi}) = H_0(\boldsymbol{I}_r + \mathbb{M}^T\boldsymbol{p}) + \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r + \mathbb{M}^T\boldsymbol{p})\cos\psi_1, \tag{6.10}$$

and the resonant Hamiltonian is cyclic in $\psi_2, \ldots, \psi_N$ and therefore the new actions $p_k, k = 2, \ldots, N$ are local integrals of motion. Recalling that we have assumed that $\|\boldsymbol{I} - \boldsymbol{I}_r\| = \|\mathbb{M}^T\boldsymbol{p}\| < \delta$, with δ "small", we Taylor expand H_0 in powers of $\mu_{kl}p_k$ up to the minimum order such that we keep the nonlinearity of the unperturbed Hamiltonian, that is up to $\mathcal{O}(\mu_{kl}^2 p_k^2)$. Since $\mathcal{O}(\mu_{kl}^2 p_k^2) \sim \mathcal{O}(\delta^2)$ we take $\mathcal{O}(\delta^2) \sim \mathcal{O}(\varepsilon)$. Therefore, again this formulation is valid within a domain of $\mathcal{O}(\sqrt{\varepsilon})$ around $\boldsymbol{I}_r$. Thus,

$$H_0(I_{rl} + p_k\mu_{kl}) \approx H_0(\boldsymbol{I}_r) + \underbrace{\left(\frac{\partial H_0}{\partial I_l}\right)_{\boldsymbol{I}_r}}_{\omega_l(\boldsymbol{I}_r)} p_k\mu_{kl} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 H_0}{\partial I_l \partial I_j}\right)_{\boldsymbol{I}_r}}_{\frac{\partial\omega_l}{\partial I_j}(\boldsymbol{I}_r)} p_k\mu_{kl}p_s\mu_{sj},$$

defining the tensor

$$\frac{1}{M_{ks}} = \mu_{kl}\left(\frac{\partial\omega_l}{\partial I_j}\right)_{\boldsymbol{I}_r}\mu_{sj},$$

and neglecting the constant term, we obtain

$$H_0(\boldsymbol{I}) \approx \omega_l(\boldsymbol{I}_r)\mu_{kl}p_k + \frac{1}{2}\frac{p_k p_s}{M_{ks}}.$$

The linear term in p_k is

$$\begin{aligned}\omega_{rl}\mu_{kl}p_k = p_1 \overbrace{\left(\mu_{11}\omega_{r1} + \mu_{12}\omega_{r2} + \cdots + \mu_{1N}\omega_{rN}\right)}^{\boldsymbol{\mu}_1\cdot\boldsymbol{\omega}_{\boldsymbol{r}}} + \\ p_2 \underbrace{\left(\mu_{21}\omega_{r1} + \mu_{22}\omega_{r2} + \cdots + \mu_{2N}\omega_{rN}\right)}_{\boldsymbol{\mu}_2\cdot\boldsymbol{\omega}_{\boldsymbol{r}}} + \\ \cdots\cdots\cdots\cdots \\ p_N \underbrace{\left(\mu_{N1}\omega_{r1} + \mu_{N2}\omega_{r2} + \cdots + \mu_{NN}\omega_{rN}\right)}_{\boldsymbol{\mu}_N\cdot\boldsymbol{\omega}_{\boldsymbol{r}}}.\end{aligned}$$

Since $\boldsymbol{\mu}_1 = \boldsymbol{m}$, $\boldsymbol{\mu}_1 \cdot \boldsymbol{\omega_r} = 0$, and thus defined the $\boldsymbol{\mu}_s$, it is $\boldsymbol{\mu}_s \cdot \boldsymbol{\omega_r} = 0$ for $s \neq 2$, while $\boldsymbol{\mu}_2 = \hat{\boldsymbol{\omega}}_r$, then the linear term reduces to

$$\omega_{rl}\mu_{kl}p_k = p_2\|\boldsymbol{\omega}_r\|.$$

Therefore, $H_0(\boldsymbol{I})$ takes the form

$$H_0(\boldsymbol{I}) \approx p_2\|\boldsymbol{\omega}_r\| + \frac{1}{2}\frac{p_k p_s}{M_{ks}}.$$

Now we should consider in (6.10) the other term that depends on the actions, $\varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r + \mathbb{M}^T\boldsymbol{p})\cos\psi_1$. As we did in the 1.5 dimensional case, since $\varepsilon\mu_{kl}p_k \sim \mathcal{O}(\varepsilon\delta) \sim \mathcal{O}(\varepsilon^{3/2})$ we just take the zero order expansion and we get for the resonant Hamiltonian

$$H_r(\boldsymbol{p}, \boldsymbol{\psi}) = p_2\|\boldsymbol{\omega}_r\| + \frac{1}{2}\frac{p_k p_s}{M_{ks}} + \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)\cos\psi_1. \tag{6.11}$$

It is important to keep in mind that the resonant Hamiltonian, only valid in D, is N-dimensional since it depends on $p_1, p_2, \ldots, p_N$ and $\psi_1, \psi_2, \ldots, \psi_N$. However as we have already mention (6.11) is cyclic in $\psi_2, \psi_3, \ldots, \psi_N$ and therefore $p_2, p_2 \ldots, p_N$ are $N-1$ local integrals of motion in D. Thus the N-dimensional Hamiltonian (6.11) reduces to a one-dimensional one. Moreover since H_r is autonomous, $H_r = h_r$ is another local integral. Therefore H_r given in (6.11) is an integrable Hamiltonian in D since it posses N-local integrals.

Now let us discus the numerical values of the local integrals $p_k, k \geq 2$. If we impose that $\boldsymbol{I}_r$ would be a point of a given trajectory of (6.11), it should be $p_k = 0, k = 2, \ldots, N$. Indeed, if we look at (6.8) and bellow (for the 3D example) if $p_k = \text{cte} \neq 0, k \geq 2$, $\boldsymbol{I}_r$ could not be a point visited by $\boldsymbol{I}$, since the motion is along the one-dimensional manifold defined by $\boldsymbol{\mu}_1 \equiv \boldsymbol{m}$. Therefore if we set $p_k = 0, k \geq 2$ in (6.11) we arrive to the following resonant Hamiltonian

$$H_r(p_1, \psi_1) = \frac{p_1^2}{2M} + \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)\cos\psi_1, \tag{6.12}$$

where

$$\frac{1}{M} = \frac{1}{M_{11}} = m_i\left(\frac{\partial\omega_i}{\partial I_j}\right)_{\boldsymbol{I}_r} m_j.$$

Therefore, the resonant Hamiltonian is a pendulum Hamiltonian for p_1 and ψ_1 where the values of the local integral $H_r = h_r$ would lead to oscillations or rotations of ψ_1. Clearly within the oscillation regime p_1 will cross the value $p_1 = 0$ in each period of oscillation.

In the one-dimensional autonomous Hamiltonian (6.12) the energy levels h_r parameterizes each solution. The rest of the $N-1$ local integrals of (6.11) are, after inverting the first of (6.8), just linear combinations of the unperturbed global integrals,

$$\tilde{\mu}_{ij}^T I_j = \text{cte}, \qquad i \neq 1,$$

where $\tilde{\mu}_{ij}^T$ is the inverse of $\mathbb{M}^T$, $\tilde{\mu}_{ik}^T \mu_{lk} = \delta_{il}$.

Let us assume that $M > 0$ and $\varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r) > 0$, then $p_1 = 0, \psi_1 = \pi$ is the stable equilibrium point, therefore for $h_r < \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)$ the system oscillates around this point. For $h_r > \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)$ the rotation regime arises and the system is not longer trapped in resonance. The separatrix corresponds to $h_r = \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)$, thus setting this level in the Hamiltonian (6.12), the equation for the separatrix is

$$\begin{aligned} H_r(p_{1s}, \psi_{1s}) &= \frac{p_{1s}^2}{2M} + \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)\cos\psi_{1s} = \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r), \\ \frac{p_{1s}^2}{2M} &= \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)(1-\cos\psi_{1s}) = 2\varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)\sin^2\frac{\psi_{1s}}{2}, \\ p_{1s} &= \pm 2\sqrt{\varepsilon M V_{\boldsymbol{m}}(\boldsymbol{I}_r)}\sin\frac{\psi_{1s}}{2}. \end{aligned}$$

Defining

$$p_r = 2\sqrt{\varepsilon M V_{\boldsymbol{m}}(\boldsymbol{I}_r)}$$

then

$$p_{1s} = \pm p_r \sin\frac{\psi_{1s}}{2}, \qquad p_r \sim \mathcal{O}(\sqrt{\varepsilon}). \tag{6.13}$$

The small oscillation frequency of ψ_1 is

$$\Omega = \sqrt{\frac{\varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)}{M}}. \tag{6.14}$$

Therefore close to the exact resonance point, p_1 oscillates around it with an amplitude of $\mathcal{O}(\sqrt{\varepsilon})$ with a slow frequency $\omega_\psi < \Omega \sim \mathcal{O}(\sqrt{\varepsilon})$, thus, in terms of Ω, p_r takes the form

$$p_r = 2M\Omega.$$

Take for instance $N = 2$ (later we will discuss this case where just two basis vectors are needed), then Fig. 5.2 is representative of the phase space structure of the resonance if we take $I \equiv I_1, I_r \equiv I_{r1}, \vartheta \equiv \vartheta_1, m \equiv m_1$ after

a section through $I_2 = \text{cte}$. Recall that in 2D, since the phase space of H_0 is foliated by two-dimensional invariant tori ($S^1 \times S^1$), I_1 should be thought as a "radial" coordinate and then the pendulum like structure should be observed for instance in a surface of section, along a circle instead of a straight line.

6.3 Resonance half-width

Let us go back to the original variables. From (6.8) and recalling that $p_2 = p_3 = \cdots = p_N = 0$ and $\boldsymbol{\mu}_1 = \boldsymbol{m}$, then

$$\boldsymbol{I} = \boldsymbol{I}_r + \boldsymbol{m} p_1$$

thus the difference $\|\boldsymbol{I} - \boldsymbol{I}_r\|$ within the oscillation regime takes its maximum value when $p_1 = p_{1s}$, the latter bounded by p_r. Therefore

$$\|\boldsymbol{I} - \boldsymbol{I}_r\|_{\max} = \|\boldsymbol{m}\| p_r.$$

Denoting $(\Delta I)_r = \|\boldsymbol{I} - \boldsymbol{I}_r\|_{\max}$, the half-width of the resonance in action space is

$$(\Delta I)_r = 2\|\boldsymbol{m}\| \sqrt{\varepsilon M V_{\boldsymbol{m}}(\boldsymbol{I}_r)},$$

after replacing p_r by (6.13). Therefore, as a difference from the non-resonant case, where the change in the unperturbed action is of order ε, the effect of a resonant perturbation is of order $\sqrt{\varepsilon}$, larger than the former.

In frequency space, take the resonant phase $\boldsymbol{m} \cdot \boldsymbol{\vartheta} = \psi_1$, so $\boldsymbol{m} \cdot \dot{\boldsymbol{\vartheta}} = \dot{\psi}_1 = p_1/M$. It is not as easy as we did in the 1.5-D system to get an explicit expression for the change of the unperturbed frequency $\boldsymbol{\omega} = \dot{\boldsymbol{\vartheta}}$. Thus we proceed in another way[2]. Let us Taylor expand $\boldsymbol{\omega}(\boldsymbol{I})$ around $\boldsymbol{I}_r$ up to first order,

$$\boldsymbol{\omega}(\boldsymbol{I}) \approx \boldsymbol{\omega}(\boldsymbol{I}_r) + \left(\frac{\partial \boldsymbol{\omega}}{\partial I_i} \right)_{\boldsymbol{I}_r} (I_i - I_{ri}),$$

with $I_i - I_{ri} = m_i p_1$, so

$$\boldsymbol{\omega}(\boldsymbol{I}) - \boldsymbol{\omega}(\boldsymbol{I}_r) \approx p_1 m_i \left(\frac{\partial \boldsymbol{\omega}}{\partial I_i} \right)_{\boldsymbol{I}_r} = p_1 \left(\boldsymbol{m} \cdot \frac{\partial}{\partial \boldsymbol{I}} \right)_{\boldsymbol{I}_r} \boldsymbol{\omega}, \tag{6.15}$$

[2]See however next Section.

and the maximum departure of $\boldsymbol{\omega}$ from $\boldsymbol{\omega}_r \equiv \boldsymbol{\omega}(\boldsymbol{I}_r)$ occurs for $p_1 = p_r$, then defining the vector $(\boldsymbol{\Delta\omega}) = (\boldsymbol{\omega} - \boldsymbol{\omega}_r)_{\max}$,

$$(\boldsymbol{\Delta\omega}) = p_r \left(\boldsymbol{m} \cdot \frac{\partial}{\partial \boldsymbol{I}} \right)_{\boldsymbol{I}_r} \boldsymbol{\omega}.$$

It is clear that $(\boldsymbol{\Delta\omega})$ thus defined in general would not have the same direction than the vector $\boldsymbol{m}$, the latter being normal to the resonant plane in frequency space. In Fig. 6.4 we represent the resonant surface $\Sigma_r^{\prime \boldsymbol{m}}$. The resonant frequency $\boldsymbol{\omega}_r \in \Sigma_r^{\prime \boldsymbol{m}}$ while $\boldsymbol{m} \perp \Sigma_r^{\prime \boldsymbol{m}}$. The vector $(\boldsymbol{\Delta\omega})$ in general does not lie in $\Sigma_r^{\prime \boldsymbol{m}}$ and is not collinear with $\boldsymbol{m}$. Then, it is natural to define the resonance-half width as the projection of $(\boldsymbol{\Delta\omega})$ in the normal direction to the resonant surface. Therefore we define the resonance-half width in frequency space as

$$(\Delta\omega)_r = (\boldsymbol{\Delta\omega}) \cdot \frac{\boldsymbol{m}}{\|\boldsymbol{m}\|} = p_r m_i \left(\frac{\partial \omega_j}{\partial I_i} \right)_{\boldsymbol{I}_r} \frac{m_j}{\|\boldsymbol{m}\|};$$

and recalling the definition of M given in (6.12), $(\Delta\omega)_r$ reduces to

$$(\Delta\omega)_r = \frac{p_r}{\|\boldsymbol{m}\| M} = 2\frac{\Omega}{\|\boldsymbol{m}\|} \propto \sqrt{\frac{\varepsilon}{M}}.$$

From the above expression for $(\Delta\omega)_r$ we conclude that if $1/M = 0$ then $(\Delta\omega)_r = 0$. This means that $(\boldsymbol{\Delta\omega})$ lies on the resonant plane and therefore the system does not leave the resonant surface. Thus the pendulum approximation does not hold and the system keeps in the exact resonance as an isochronous oscillator. Therefore the condition $1/M \neq 0$ leads to

$$\frac{1}{M} = m_i \left(\frac{\partial \omega_i}{\partial I_j} \right)_{\boldsymbol{I}_r} m_j \neq 0,$$

which is different from the former assumption

$$\left(\frac{\partial \omega_i}{\partial I_j} \right)_{\boldsymbol{I}_r} \neq 0,$$

which ensures that $\boldsymbol{\omega}(\boldsymbol{I})$ is a one-to-one application. In fact $1/M \neq 0$ is a more restricted condition than the later. The above condition for the applicability of the pendulum model can be stated in another way. Indeed,

$$\frac{1}{M} = m_i \left(\frac{\partial \omega_i}{\partial I_j} \right)_{\boldsymbol{I}_r} m_j = \boldsymbol{m} \cdot \boldsymbol{n}_r \neq 0,$$

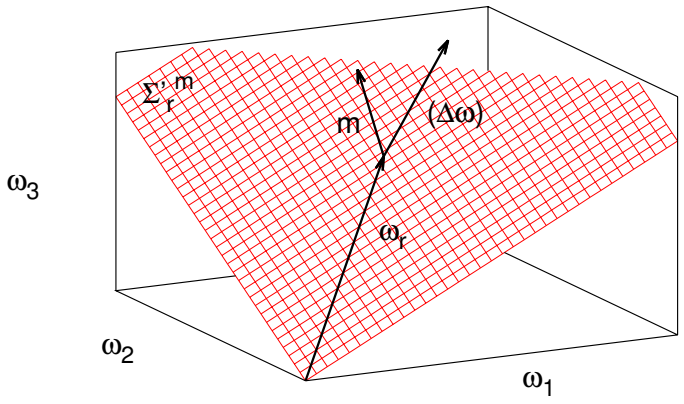

Figure 6.4: Vectors $\boldsymbol{m}$ and $(\boldsymbol{\Delta\omega})$ at the resonant point $\boldsymbol{\omega}_r$. The resonant vector $\boldsymbol{m} \perp \Sigma'^{\boldsymbol{m}}_r$, the resonant plane, $\boldsymbol{\omega}_r \in \Sigma'^{\boldsymbol{m}}_r$ while $(\boldsymbol{\Delta\omega})$ is not normal to $\Sigma'^{\boldsymbol{m}}_r$ and should be $(\boldsymbol{\Delta\omega}) \notin \Sigma'^{\boldsymbol{m}}_r$ in order to the pendulum model for the non-linear resonance works.

that is, since the vector $\boldsymbol{e}_s \in \Pi_r$ is defined such that $\boldsymbol{e}_s \cdot \boldsymbol{n}_r = 0$, then the condition $\boldsymbol{m} \cdot \boldsymbol{n}_r \neq 0$ means that $\boldsymbol{m} \in \Pi_r$ should not be collinear to $\boldsymbol{e}_s$. If this occurs $\hat{\mathcal{B}}$ is not a well defined basis since $\boldsymbol{m}$ and $\boldsymbol{e}_s$ are no longer linearly independent.

6.4 Motion in the original variables

As we have already seen, the original action changes as (6.8)

$$\boldsymbol{I} = \boldsymbol{I}_r + \boldsymbol{m}p_1,$$

where p_1 is the pendulum momentum that changes with time accordingly to (6.12). Now, let us look for the variation of the new phases, ψ_k. From (6.11) we get

$$\dot{\psi}_l = \frac{\partial H_r}{\partial p_l} = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{p_s\delta_{kl}}{2M_{ks}} + \frac{p_k\delta_{sl}}{2M_{ks}} = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{p_s}{2M_{ls}} + \frac{p_k}{2M_{kl}},$$

since k, s are dummy indexes of sum we can write

$$\dot{\psi}_l = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{p_s}{2M_{ls}} + \frac{p_s}{2M_{sl}},$$

and by definition

$$\frac{1}{M_{ls}} = \mu_{li}\left(\frac{\partial\omega_i}{\partial I_j}\right)_{\boldsymbol{I}_r}\mu_{sj},$$

is symmetric[3], it results

$$\dot{\psi}_l = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{p_s}{M_{ls}}.$$

Since the actions p_s are such that for $s \geq 2, p_s = 0$ then the sum over s reduces to only one term just for $s = 1$ and then

$$\dot{\psi}_l = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{p_1}{M_{l1}}.$$

From (6.12),

$$\dot{\psi}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{M}, \quad \rightarrow \quad p_1 = M\dot{\psi}_1$$

and then we get for $\dot{\psi}_l$,

$$\dot{\psi}_l = \|\boldsymbol{\omega}_r\|\delta_{2l} + \frac{M}{M_{l1}}\dot{\psi}_1.$$

From the above equation we note that although the $p_l, l \geq 2$ are local integrals of motion, the frequencies $\dot{\psi}_l$ do change with time through $\dot{\psi}_1$ and are not local integrals. The above expression can be immediately integrated to get

$$\psi_l(t) = \|\boldsymbol{\omega}_r\|\delta_{2l}t + \frac{M}{M_{l1}}\psi_1(t) + \psi_l^0,$$

showing that the oscillations in ψ_1 influence the time evolution of all the new phases.

Let us go back to the original angles. From (6.8)

$$\boldsymbol{\psi} = \mathbb{M}\boldsymbol{\vartheta}, \qquad \text{or} \qquad \psi_k = \mu_{kl}\vartheta_l.$$

[3] $M_{ls} = M_{sl}$ since the constant factor $(\partial\omega_i/\partial I_j)_{\boldsymbol{I}_r}$ is multiplied by μ_{li} and μ_{sj}, so exchanging l by s we get the very same sum over i and j.

If we introduce $\mathbb{M}^{-1} = \{\tilde{\mu}_{lk}\}$ the inverse of $\mathbb{M}$:

$$\tilde{\mu}_{lk}\mu_{ks} = \mu_{kl}\tilde{\mu}_{ls} = \delta_{ks},$$

then if we multiply from the left ψ_k by $\tilde{\mu}_{sk}$ we get

$$\tilde{\mu}_{sk}\psi_k = \underbrace{\tilde{\mu}_{sk}\mu_{kl}}_{\delta_{sl}}\vartheta_l, \qquad \rightarrow \qquad \tilde{\mu}_{sk}\psi_k = \vartheta_s,$$

therefore introducing the solution for $\psi_k(t)$ given above,

$$\vartheta_s = \tilde{\mu}_{sk}\left(\|\boldsymbol{\omega}_r\|\delta_{2k}t + \frac{M}{M_{k1}}\psi_1(t) + \psi_k^0\right) = \underbrace{\|\boldsymbol{\omega}_r\|\tilde{\mu}_{sk}\delta_{2k}}_{b_s}\,t + \underbrace{\tilde{\mu}_{sk}\frac{M}{M_{k1}}}_{a_s}\,\psi_1(t) + \underbrace{\tilde{\mu}_{sk}\psi_k^0}_{\vartheta_s^0}.$$

Let us compute then the vector components a_s and b_s recalling that

$$\frac{1}{M_{k1}} = \mu_{ki}\left(\frac{\partial\omega_i}{\partial I_j}\right)_{\boldsymbol{I}_r}\mu_{1j}, \qquad \mu_{1j} = m_j, \qquad \mu_{2j} = \hat{\omega}_{rj}$$

then

$$a_s = \tilde{\mu}_{sk}\frac{M}{M_{k1}} = \tilde{\mu}_{sk}\mu_{ki}\left(\frac{\partial\omega_i}{\partial I_j}\right)_{\boldsymbol{I}_r}\mu_{1j}M = \delta_{si}\left(\frac{\partial\omega_i}{\partial I_j}\right)_{\boldsymbol{I}_r}m_jM = \left(\frac{\partial\omega_s}{\partial I_j}\right)_{\boldsymbol{I}_r}m_jM.$$

For b_s we have

$$b_s = \|\boldsymbol{\omega}_r\|\tilde{\mu}_{sk}\delta_{2k} = \|\boldsymbol{\omega}_r\|\tilde{\mu}_{s2},$$

and compute $\mu_{is}b_s$,

$$\mu_{is}b_s = \|\boldsymbol{\omega}_r\|\mu_{is}\tilde{\mu}_{s2} = \|\boldsymbol{\omega}_r\|\delta_{i2} \quad \rightarrow \quad \mu_{2s}b_s = \|\boldsymbol{\omega}_r\| \quad \rightarrow \quad \hat{\omega}_{rs}b_s = \|\boldsymbol{\omega}_r\|,$$

and since $\hat{\omega}_{rs} = \omega_{rs}/\|\boldsymbol{\omega}_r\|$ we obtain

$$\omega_{rs}b_s = \|\boldsymbol{\omega}_r\|^2 \qquad \rightarrow \qquad b_s = \omega_{rs}.$$

Thus we finally get for ϑ_s

$$\vartheta_s = \omega_{rs}t + \left(\frac{\partial\omega_s}{\partial I_j}\right)_{\boldsymbol{I}_r}m_jM\psi_1(t) + \vartheta_s^0, \tag{6.16}$$

where $\psi_1(t)$ is the solution of the resonant Hamiltonian (6.12). If $\varepsilon = 0$, $H_r(p_1, \psi_1) = p_1^2/2M$ and then all the $p_i, i = 1, \ldots, N$ are constant with $p_1 = 0$ ($H_r = 0$) since $\boldsymbol{I} = \boldsymbol{I}_r$ and then we set $\psi_1 = 0$[4]. Therefore (6.16) reduces to

$$\vartheta_s = \omega_{rs} t + \vartheta_s^0, \qquad \varepsilon = 0,$$

which is in fact the solution for $H_0(\boldsymbol{I})$.

From (6.16) we can easily compute how the frequency changes due to the resonant perturbation, just taking the time-derivative of (6.16) leading to

$$\omega_s = \omega_{rs} + \left(\frac{\partial \omega_s}{\partial I_j}\right)_{\boldsymbol{I}_r} m_j M \dot{\psi}_1(t)$$

where from (6.12) $\dot{\psi}_1 = p_1/M$, then

$$\omega_s = \omega_{rs} + \left(\frac{\partial \omega_s}{\partial I_j}\right)_{\boldsymbol{I}_r} m_j p_1(t),$$

which in its vector form reads

$$\boldsymbol{\omega} = \boldsymbol{\omega}_r + p_1(t) \left(\boldsymbol{m} \cdot \frac{\partial}{\partial \boldsymbol{I}}\right)_{\boldsymbol{I}_r} \boldsymbol{\omega},$$

which completely agree with (6.15).

In Fig. 6.5 we illustrate the dynamics for $N = 2$ in action space. This case, though eloquent, is not the best one since only two vectors are needed to define the new basis $\hat{\mathcal{B}}$, just $\boldsymbol{m}$ and $\boldsymbol{\omega}_r$. However it is instructive to see how the motion proceeds along $\boldsymbol{m}$ measured by the local action p_1. The figure clearly shows that under a resonant perturbation, the resonance become in fact a resonance layer of width $(\Delta I)_r$.

As we have already mentioned, the non-linear stabilization due to a resonant perturbation depends on M. As a deference from the 1.5 degree of freedom model, in a multidimensional system it could be $1/M = 0$. In such case $(\boldsymbol{\Delta\omega})$ lies on the resonant plane, the non-linear system keeps in resonance for ever as a linear one. We can say then that if this occurs the non-linear stabilization does not work any longer. Note that $1/M$ depends only on the

[4]More generally, $\psi_1 = \psi_1^0 =$ cte so taking $\psi_1 \neq 0$ just implies a shift in $\vartheta_s^0 \to \vartheta_s^0 + \left(\frac{\partial \omega_s}{\partial I_j}\right)_{\boldsymbol{I}_r} m_j M \psi_1^0$ that though it has no any consequence in the dynamics implies an artificial change of the initial conditions in H_0.

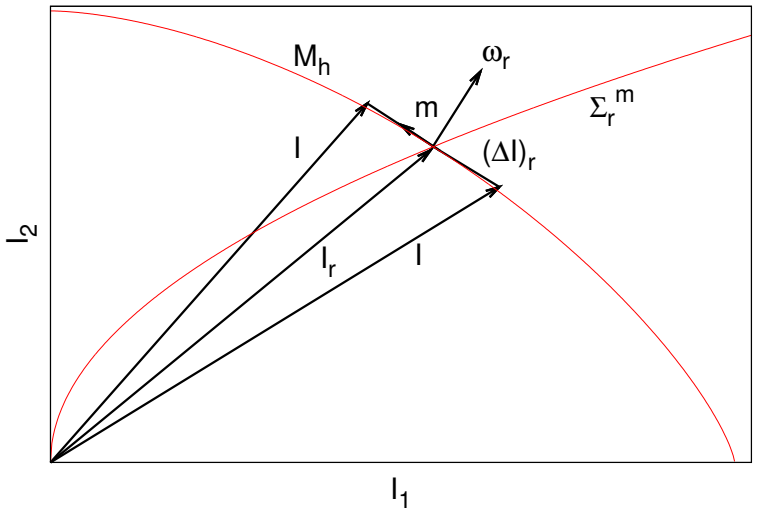

Figure 6.5: Illustration of the resonant motion for 2D in action space. The resonant action corresponds to the intersection of M_h and $\Sigma_r^{\boldsymbol{m}}$, the energy and resonant curves respectively. The motion, p_1, is along $\boldsymbol{m}$, tangent to M_h at $\boldsymbol{I}_r$. The frequency vector $\boldsymbol{\omega}_r \perp M_h$.

unperturbed Hamiltonian, thus the non-linear stabilization strongly depends on H_0. However, to be precise, the condition $1/M = 0$ is a necessary but not sufficient condition for the so-called *quasi-isochronism*, since the stabilization could be due to higher order terms in p_s of the expansion of H_0.

Finally, let us mention that we have always drawn *convex* energy surfaces, in which every point in M_h has a single point of tangency with the tangent plane at the very same point. If M_h is not convex the tangent plane could intersect the latter in some submanifold. The convexity of M_h is in fact a requirement to the pendulum model of a non-linear resonance works, however this discussion is, by now, out of the scope of the present chapter.

Chapter 7

Resonance interaction I

The example discussed in the last chapter is quite restricted, since only one resonant term is retained in the perturbation. In order to describe a more realistic situation, it seems reasonable to include some other harmonics of the perturbation (i.e., several vectors $\boldsymbol{m}$). Depending on the initial conditions, these additional terms may be killed by the application of the averaging method, but in this chapter we will consider the opposite case, when the system is moving within a region of the phase space where two or more resonances are present. Any attempt to describe the *resonance interaction* in a rigorous way is out of the scope of the present discussion. Nevertheless, Chirikov developed a qualitative criterion called *overlap of non–linear resonances*, that may help us to understand the motion under the influence of several resonances.

7.1 The motion under the presence of two resonances

Let us consider the Hamiltonian (6.1) and a given open set $\bar{D} \subset \mathbb{R}^N$ of size ξ small, such that for two independent harmonics $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ it is

$$\|\boldsymbol{m}^1 \cdot \boldsymbol{\omega}(\boldsymbol{I})\| \ll 1, \qquad \|\boldsymbol{m}^2 \cdot \boldsymbol{\omega}(\boldsymbol{I})\| \ll 1. \tag{7.1}$$

The above condition means that the system is close to two different resonances,

$$\boldsymbol{m}^1 \cdot \boldsymbol{\omega}(\boldsymbol{I}_r^1) = 0, \qquad \boldsymbol{m}^2 \cdot \boldsymbol{\omega}(\boldsymbol{I}_r^2) = 0, \qquad \boldsymbol{I}_r^1, \boldsymbol{I}_r^2 \in \bar{D}, \qquad \|\boldsymbol{I}_r^1 - \boldsymbol{I}_r^2\| \ll \xi.$$

We wonder how the motion looks like under the effect of more than a single non-linear resonance when $N > 2$. Assume that in $\bar{D}$, only the harmonics $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ in the Fourier expansion are slow, thus we can average out the remainder terms and considering the real part of the series, we get for $\boldsymbol{I} \in \bar{D}$,

$$H(\boldsymbol{I}, \boldsymbol{\vartheta}) = H_0(\boldsymbol{I}) + \varepsilon V_{\boldsymbol{m}^1}(\boldsymbol{I}) \cos(\boldsymbol{m}^1 \cdot \boldsymbol{\vartheta}) + V_{\boldsymbol{m}^2}(\boldsymbol{I}) \cos(\boldsymbol{m}^2 \cdot \boldsymbol{\vartheta}). \tag{7.2}$$

The variation of the unperturbed integral $\boldsymbol{I}$ due to the presence of the perturbation is now

$$\dot{\boldsymbol{I}} = -\frac{\partial H}{\partial \boldsymbol{\vartheta}} = \varepsilon \boldsymbol{m^1} V_{\boldsymbol{m}^1}(\boldsymbol{I}) \sin(\boldsymbol{m}^1 \cdot \boldsymbol{\vartheta}) + \varepsilon \boldsymbol{m^2} V_{\boldsymbol{m}^2}(\boldsymbol{I}) \sin(\boldsymbol{m}^2 \cdot \boldsymbol{\vartheta}).$$

Then under the effect of two resonant perturbing terms the change of the unperturbed action, $\Delta \boldsymbol{I}$, is confined to a two-dimensional manifold. Indeed $\Delta \boldsymbol{I}$ has two components, one in the direction of $\boldsymbol{m^1}$ and other in $\boldsymbol{m^2}$ direction. It is clear then that we cannot apply all the geometrical arguments used in the previous chapter (for a single resonance) to construct a new basis, since now there are two resonance conditions. However it is clear that the motion would be confined to the two dimensional manifold expanded by $\boldsymbol{m^1}$ and $\boldsymbol{m^2}$.

Thus we should try another way. Let us briefly discuss how to proceed. Imagine we could still define a new basis at $\boldsymbol{I}_r \approx \boldsymbol{I}_r^1 \approx \boldsymbol{I}_r^2$, $\hat{\mathcal{B}} = \{\boldsymbol{\mu}_1 = \boldsymbol{m}^1, \boldsymbol{\mu}_2 = \boldsymbol{m}^2, \boldsymbol{\mu}_3, \ldots, \boldsymbol{\mu}_N\}$ such that p_1, p_2 are the components of the action $\boldsymbol{p}$ in the direction of $\boldsymbol{m^1}$ and $\boldsymbol{m^2}$, respectively. Clearly this assumption is not true in general because $\boldsymbol{p}$ is not well defined since it should locally describe the motion in some neighborhood around the exact resonant point, $\boldsymbol{I}_r^1$ or $\boldsymbol{I}_r^2$, and therefore it becomes ambiguous where $\hat{\mathcal{B}}$ is defined[1]. Anyway, let us go on and suppose that a canonical transformation similar to the one introduced in (6.8), can be performed so $\psi_k = \mu_{kl}\vartheta_l$, where $\mu_{kl} = (\boldsymbol{\mu}_k)_l$. Then $\psi_1 = \boldsymbol{m}^1 \cdot \boldsymbol{\vartheta}, \psi_2 = \boldsymbol{m}^2 \cdot \boldsymbol{\vartheta}$ and therefore the Hamiltonian becomes cyclic in $\psi_3, \ldots, \psi_N$, thus the actions $p_3, \ldots, p_N$ are local integrals of motion and the N-dimensional Hamiltonian (7.2) reduces to a 2-dimensional one, $H_{rr}(p_1, p_2, \psi_1, \psi_2)$. Being the latter autonomous we know that $H_{rr} = h_{rr}$ is a local integral. But in general, it would not be possible to find any additional (local) integral, so $H_{rr}(p_1, p_2, \psi_1, \psi_2)$ is by rule non-integrable since it has two degrees of freedom and only one integral exists. Therefore, we should deal with another approach to the problem.

[1]This new basis would be well defined when $\boldsymbol{I}_r^1 = \boldsymbol{I}_r^2$, that is when the system is in an exact double resonance.

7.2 The overlap criterion

Let us left aside any attempt to give a mathematical formulation of the motion under the influence of two resonances and just provide heuristic arguments about the nature of the motion in $\bar{D}$.

Each resonance will determine its own domain in the phase space, but the motion in the vicinity of one resonance will be affected by the presence of the other. If the resonances are located "far enough"to each other, we may expect the motion to be confined to the neighborhood of one resonance or the other, depending on the initial conditions. The picture of a slight distortion of the pendulum model is then a fair approximation to the actual motion, and to give a qualitative criterion, we can neglect the effect of the perturbing resonance. Thus, each resonance has its own pendulum model, with small oscillations, separatrix and rotation about the resonant value $\boldsymbol{I}_r^1$ and $\boldsymbol{I}_r^2$ as Fig. 7.1 illustrates.

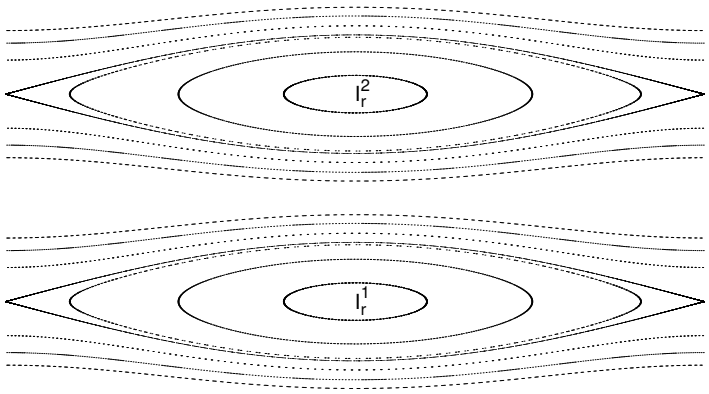

Figure 7.1: Sketch of the resonance interaction when just two resonances are considered in $\bar{D}$. Each resonance is described by a simple pendulum model around $\boldsymbol{I}_r^1$ and $\boldsymbol{I}_r^2$.

On the other hand, if the two resonances are "close enough" to each other, then it is reasonable to expect the motion not to be confined within one domain, and the trajectory may jump from one resonance domain to the other; i.e., the action could range from some neighborhood of $\boldsymbol{I}_r^1$ to some

other neighborhood of $\boldsymbol{I}_r^2$. This kind of motion seems to have nothing to do with any significant instability (large variation of the unperturbed action $\boldsymbol{I}$) since $\boldsymbol{I}_r^1 \approx \boldsymbol{I}_r^2$. But, as shown in many numerical experiments (see next Chapter), the motion becomes irregular as if the system were dominated by a stochastic force. Nevertheless the Hamiltonian (7.2) does not include any stochastic interaction like, for example, the Brownian motion. This is the reason why the motion in question was called *stochastic instability* or nowadays *chaotic*.

From these intuitive considerations we may infer that a plausible condition for the stochastic instability is that the separation between the resonances (in action or frequency space), is of the order of the resonance width; i.e., an overlap of resonances. Let us put it in another way, the overlap of resonances takes place when the unperturbed separatrix of one resonance touches the other. Mathematically speaking, when the unperturbed separatrix of two different resonances intersect each other, this intersection is called *heteroclinic intersection*[2].

Let us consider, for instance, the frequency space. Since $\boldsymbol{m}^1 \cdot \boldsymbol{\omega} \approx 0$ and $\boldsymbol{m}^2 \cdot \boldsymbol{\omega} \approx 0$; i.e., the initial conditions are chosen in such a way that $\boldsymbol{\omega}$ is close to any of the two fixed vectors $\boldsymbol{\omega}_r^1$ and $\boldsymbol{\omega}_r^2$, each of them belonging to its own resonance plane. Let $\|\boldsymbol{\omega}_r^1 - \boldsymbol{\omega}_r^2\| = \Delta$. The resonances have a frequency width $(\Delta\omega)_r^1$ and $(\Delta\omega)_r^2$ respectively. Thus, the condition for the overlap of resonances may be formulated as

$$\|(\Delta\omega)_r^1 + (\Delta\omega)_r^2\| \approx \Delta,$$

where $\|\cdot\|$ denotes the "length" measured over the energy surface (see Fig. 7.2 for a given example). As $(\Delta\omega)_r^i \sim \sqrt{\varepsilon}$, then the latter condition gives an estimate of the so–called *stability border*. This means that if ε_c is the value of the perturbation parameter that satisfies the overlap condition, then for $\varepsilon < \varepsilon_c$ we may expect the system to be confined within the domain of one resonance. That is, the motion is stable, as described in the previous chapter. On the other hand, if $\varepsilon \gtrsim \varepsilon_c$, then the stochastic instability arises; the resonances are connected and the motion proceeds over both domains.

Let us take $N = 2$. The resonance condition leads, in the frequency space, to lines with different slopes (given by $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$) passing through

[2]An *homoclinic intersection* takes place when both separatrix correspond to the very same resonance, as for instance in the pendulum where the unstable and stable manifolds match exactly, but under a rather small perturbation, they split and the intersection angle between them is different from zero.

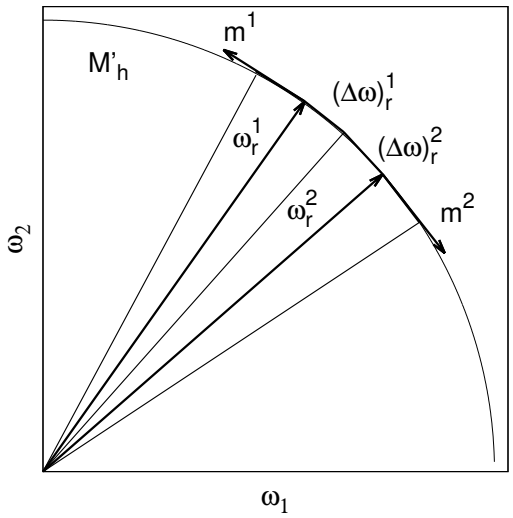

Figure 7.2: Sketch of the condition for the overlap of resonances. Each resonance is defined by the vectors $\boldsymbol{\omega}_r^1$ and $\boldsymbol{\omega}_r^2$ at the point of intersection of the energy surface M_h' and the resonant surfaces (lines in this case). The resonant vectors $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ are orthogonal to $\boldsymbol{\omega}_r^1$ and $\boldsymbol{\omega}_r^2$ respectively. Each resonance has its own width $(\Delta\omega)_r^i \propto \sqrt{\varepsilon}$. Here it is represented the situation in which $(\Delta\omega)_r^1 + (\Delta\omega)_r^2 = \Delta \equiv \|\boldsymbol{\omega}_r^1 - \boldsymbol{\omega}_r^2\|$, the overlap condition.

the origin. The energy surfaces are certain convex curves. Then, fixing the energy, the system is confined to that curve. The resonant (fixed) values $\boldsymbol{\omega}_r^1$ and $\boldsymbol{\omega}_r^2$ are the intersection points of the latter curve with the resonant lines. Fig. 7.2 represents this geometry for a particular case of overlap resonances (see below). By assumption, the initial conditions are such that $\boldsymbol{\omega}_r^1 \approx \boldsymbol{\omega}_r^2 \approx \boldsymbol{\omega}_r$. Then, the separation between the two resonances is

$$\Delta \approx \|\boldsymbol{\omega}^r\| \frac{\|\boldsymbol{m}^1 \times \boldsymbol{m}^2\|}{\|\boldsymbol{m}^1\| \|\boldsymbol{m}^2\|},$$

which is a function of the energy through $\|\boldsymbol{\omega}^r\|$. The above derivation rests on the fact that the angle between $\boldsymbol{\omega}_r^1$ and $\boldsymbol{\omega}_r^2$ is equal to the angle between their normals, $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$. On the other hand, the frequency width may be

put in the form

$$(\Delta\omega)_r^i = \delta(\boldsymbol{m}^i)\sqrt{\epsilon}, \qquad \delta(\boldsymbol{m}) = \frac{2}{\|\boldsymbol{m}\|}\sqrt{\frac{|V_{\boldsymbol{m}}|}{|M|}}.$$

Then the stability border is given by:

$$(\delta(\boldsymbol{m}^1) + \delta(\boldsymbol{m}^2))\sqrt{\varepsilon}_{\rm c} \approx \Delta, \qquad \text{or}$$

$$\varepsilon_{\rm c} \approx \frac{\Delta^2\|\boldsymbol{m}^1\|^2\|\boldsymbol{m}^2\|^2}{4\left(\|\boldsymbol{m}^2\|^2\sqrt{V_{\boldsymbol{m}^1}/M^1} + \|\boldsymbol{m}^1\|^2\sqrt{V_{\boldsymbol{m}^2}/M^2}\right)^2}.$$

Therefore, for $\varepsilon < \varepsilon_{\rm c}$ we can ensure the stability of the motion since, for very small perturbations, the size of each resonance is rather small and their domains do not overlap[3]. But as the perturbation increases, the domains become larger and they may overlap, leading to a connected region of stochastic motion.

If we consider more than two resonances, the qualitative picture is similar. In such a case, since the overlap includes several resonances, the connected region for the motion becomes larger and the unperturbed integrals may experience large variations, i.e., we have a gross instability.

From the beginning we have not considered the case of *intersection of resonance surfaces* or *multiple resonances* which is not a simple problem but we may infer that a stochastic, chaotic domain appears in the neighborhood of the intersection of both surfaces, due to the overlapping of resonance domains.

As we shall see in a forthcoming chapter, for $\varepsilon < \varepsilon_{\rm c}$ the main effect of the interaction of two resonances is to produce a qualitative change in the separatrix of the unperturbed resonance. This smooth curve becomes a layer, the so–called *stochastic layer*, since a stochastic behavior appears in the vicinity of the separatrix. All the invariant curves in a neighborhood of the separatrix disappear and, instead, a stochastic motion proceeds *across* the layer (i.e. in the direction of p_1, following the notation of the single resonance description) of finite width. This kind of motion is quite different from that of

[3]This statement is only true for the first order resonances. If we consider resonances at all orders in ε many higher order resonances appear. Therefore, the overlap criterion for the first order resonances is qualitatively sufficient but not necessary condition for the stability of the motion.

an isolated resonance since, in the latter case, the motion follows a smooth curve in the (p_1, ψ_1) space. The stochastic layer is located at the edge of the resonant layer, then oscillations and rotations for the pendulum model are actually separated by a region of irregular, chaotic motion. Though the overlap criterion misses this fact as well as many other aspects, the results given by this simple and intuitive approach are in good agreement with several numerical simulations. In fact it can be shown that, in general, the overlap criterion provides a stability border that is of the same order as that obtained using a rigorous mathematical approach by the so-called KAM theory, that we will outline later. As we shall discuss at the end of this notes, the existence of a stochastic layer and the intersection of resonance surfaces when $N > 2$ (see Fig. 7.3) is the key point in the discussion of a distinctive property of multidimensional Hamiltonian systems, the so-called (controversial) *Arnold diffusion*.

The example considered above to derive ε_c is for a two-dimensional autonomous Hamiltonian system. For these systems, the resonance lines do not intersect (except at the origin). Therefore, the unique way to obtain a connected domain of stochastic motion is through an overlap of resonances. In other words, the motion becomes unstable, stochastic, for certain values of the perturbation parameter. This is just a consequence of the topology of the phase space for $N = 2$. Indeed, we may say that the 2-dimensional tori divide the 3-dimensional energy surface. This fact implies that any transition from one torus to another is only possible through all the intermediate tori between them. If $\varepsilon \lesssim \varepsilon_c$, most of the torus structure survives, leading to a picture similar to that for an integrable Hamiltonian. The tori act as a barrier for the stochastic motion and confine it to the stochastic layers. If we project the motion in question onto the action or frequency space we have, as we mentioned above, an energy surface which is a curve where every point on it represents a different ratio for the components of the frequency vector (see for instance Figs. 6.5 and 7.2). A rational ratio corresponds to a resonance while an irrational one corresponds to a non-resonant orbit. If the system is confined within a resonance, it will move then along the energy curve and away from the resonant value by a distance which is about the amplitude of oscillation around this point ($p_r \sim \sqrt{\varepsilon}$). Although an instability is always present close to the separatrix (the stochastic layer), this region of stochastic motion is rather small, its width being exponentially small with the separation between interacting resonances (see next chapters). Therefore we conclude that, for $N = 2$ and $\varepsilon < \varepsilon_c$, the stability of the motion, in a

broad sense, is preserved.

The story is quite different for $N > 2$. It is not possible to guarantee the confinement of the system to a neighborhood of the resonant value. The tori no longer divide the energy surface; the resonant surfaces do intersect everywhere on the energy surface, leading to a network over it. Thus, an instability may occur, even for very small values of the perturbation. This peculiar instability, discovered by Arnold in 1964 in a purely mathematical paper, seems to be a universal one, since it is always present despite the smallness of the perturbation if $N > 2$. In order to give a picture of this geometry, let us consider the unperturbed Hamiltonian $H_0 = \|\boldsymbol{I}\|^2/2$, where $\boldsymbol{I}$ is a 3–dimensional vector, and a perturbation given by the full Fourier expansion with $\varepsilon < \varepsilon_c$. In this case $\boldsymbol{\omega}(\boldsymbol{I}) = \boldsymbol{I}$ and the action and frequency spaces are the same. The energy surfaces are concentric spheres. Since the resonant surfaces are planes passing through the origin, we see that the intersection of any resonant plane with a given energy surface is a great circle. It is clear that these great circles intersect over the whole surface of the sphere leading to the so–called *Arnold Web*. In Fig 7.3 we illustrate all this but just for the unperturbed Hamiltonian H_0. Under the effect of the perturbation each great circle becomes a layer of width $\sim \sqrt{\varepsilon}$. Moreover, at the edges of this layers, a small domain of chaotic motion would appear (the stochastic layer–see next chapter).

The existence of this web does not depend on the strength of the perturbation. Let us fix the energy and consider initial conditions very close to a given resonance, where the motion is confined. Just to distinguish the actual resonance from the rest of the perturbing (non-resonant) terms, let us call it *guiding resonance*. Certainly, for other initial conditions, one of the perturbing terms may become the guiding resonance, while the former resonance will then play the same role as the rest of the perturbing terms. The guiding resonance surface leads, on the energy surface, to a certain great circle (in fact, to a resonant layer). The conservation of energy confines the motion to the sphere and, the resonance condition, restricts it to the resonant layer of the guiding resonance. Without perturbation (i.e, keeping only H_0 and the resonant term) the motion proceeds then over the tangent plane to the energy surface (at the resonant value) in the direction of the guiding resonant vector, the latter being normal to the resonance plane. Therefore, if the initial conditions are chosen near the separatrix of the resonance, then the presence of perturbing terms (besides the guiding resonant one) gives rise to a motion *across* the resonant layer of the guiding resonance ("transversal

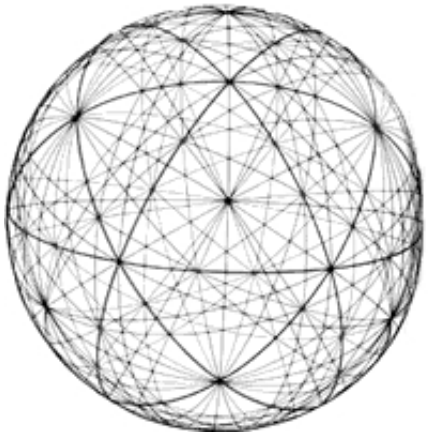

Figure 7.3: Sketch of the Arnold web for $H_0 = \boldsymbol{I}^2/2$ and several different intersections of the resonant surfaces with the energy surface. Motion along the "guiding" resonance means along the layers of the great circles. The bold curves may represent the motion along the different guiding resonances. In fact this motion may occur along the stochastic layer at the edges of the resonance layer instead of the center of the resonance. Note the difference between the two dimensional case illustrated in Fig. 7.2 for the frequency space. Figure taken from the web.

to the great circle") and modifies the edges of the latter leading to the appearance of the stochastic layer. It follows then that close to the resonant value, the great circle looks like a spherical layer where the motion within its edges is stochastic. However, the energy and resonance constraints also allow for a motion in the remainder direction ("along the great circle", the direction of the basis vector $\boldsymbol{e}$). We expect that the perturbation may also drive the stochastic motion *along* the stochastic layer of the guiding resonance. Following the notation of the previous chapter by motion along the stochastic layer we essentially mean that the components of the action, p_k, $k \neq 1$, will change with time, since now they are not exact integrals due to the dependence of the perturbation on several phases. During some time the stochastic motion may proceed along the stochastic layer of the guiding resonance. When the motion along this layer reaches a point of intersection with some other great circle (other resonant surface), a stochastic domain around the latter also appears, due to the overlap of both resonances. Although it has not been mathematically proved, it might be possible that the motion

proceed now *along* this second layer: a new guiding resonance. Since the set of resonances do intersect over the whole energy surface, we conclude then that the Arnold web is actually a network of layers where the motion within it is stochastic. On the other hand, the motion outside the web is regular and stable. If the initial conditions are chosen within any stochastic layer, then the stochastic motion might spread over the whole web through the intersecting points. From these qualitative considerations we then infer that it could be possible the connection of all regions on the energy surface where the motion is chaotic. Therefore the properties of the stochastic component could be the same over the whole phase space accessible to the system.

As we have already said, for $N = 2$ and if the perturbation is small enough, the variation of the unperturbed integrals is rather small, just confined to the stochastic layer. Motion "along" the layer does not exist due to the dimensionality (two) of the action or frequency space. But, for $N > 2$, we have at least one more degree of freedom where the motion may proceed. In the latter case, large variations of the unperturbed integrals might occur. As it is assumed that the stochastic motion admits a diffusion-like description (which is not always the case) this instability was called Arnold diffusion (though Arnold had never used this word in his celebrated paper). These questions will be addressed in the forthcoming chapters.

7.3 A simple example

Though we have presented the overlap criterion for two coupling resonances, the latter clearly could be applied in case of an external perturbation. Thus, in order to illustrate this other case, let us consider a one dimensional quartic oscillator and a periodic external perturbation that depends on two independent frequencies Ω_1 and Ω_2;

$$\begin{aligned} H(p,x,t) &= \frac{p^2}{2} + \frac{x^4}{4} + \varepsilon x(f_1 \cos\tau_1 + f_2 \cos\tau_2), \\ \tau_1 &= \Omega_1 t + \tau_1^0, \qquad \tau_2 = \Omega_2 t + \tau_2^0 \end{aligned} \tag{7.3}$$

where f_1 and f_2 are constant amplitudes and ε small. The unperturbed Hamiltonian is then

$$H_0 = \frac{p^2}{2} + \frac{x^4}{4},$$

the quartic oscillator, while

$$\varepsilon V = \epsilon x(f_1 \cos\tau_1 + f_2 \cos\tau_2)$$

is the perturbation.

We perform a canonical transformation $(p, x) \to (I, \vartheta)$ in H_0 such that (see Section 2.3)

$$H_0(I) = AI^{4/3},$$

where $A = (3\beta/2\sqrt{2})^{4/3}$. Moreover, in Section 2.3 we have already found that

$$x(I, \vartheta) \approx a(I)\cos\vartheta, \qquad a(I) = CI^{1/3},$$

is a fair approximate solution for the motion, with $C \approx 1$ a numerical constant and $\vartheta = \omega t$, where the nonlinear frequency is

$$\omega = \beta a = \sqrt{2}\beta h^{1/4} = \frac{4A}{3} I^{1/3}.$$

In terms of action-angle variables the perturbation reads,

$$\varepsilon V \approx \varepsilon a(I)\cos\vartheta(f_1 \cos\tau_1 + f_2 \cos\tau_2),$$

using the trigonometric identity

$$\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2},$$

it reduces to

$$\varepsilon V \approx \varepsilon \frac{a(I)}{2} \left\{ f_1(\cos(\vartheta + \tau_1) + \cos(\vartheta - \tau_1)) + f_2(\cos(\vartheta + \tau_2) + \cos(\vartheta - \tau_2)) \right\}.$$

Assume that all initial conditions (for instance $\Omega_1, \Omega_2 > 0, \omega(h) > 0$) are such that $(\vartheta - \tau_1)$ and $(\vartheta - \tau_2)$ are slow with respect to $(\vartheta + \tau_1)$ and $(\vartheta + \tau_2)$. Then we can average out the fast phases keeping in the perturbation only the slow ones. Thus

$$\varepsilon V \approx \varepsilon \frac{a(I)}{2} \left\{ f_1 \cos(\vartheta - \tau_1) + f_2 \cos(\vartheta - \tau_2) \right\}.$$

Therefore the full Hamiltonian (7.3) in action-angle variables and keeping only the slow phases is

$$H(I, \vartheta, \tau_1, \tau_2) = AI^{4/3} + \varepsilon \frac{a(I)}{2} \left\{ f_1 \cos(\vartheta - \tau_1) + f_2 \cos(\vartheta - \tau_2) \right\}. \qquad (7.4)$$

Depending on the energy level $H_0(I) = h$ and the difference $|\Omega_1 - \Omega_2|$, each term of the perturbation would be resonant within a domain of $\mathcal{O}(\sqrt{\varepsilon})$ around the resonant value h_{ri} of I_{ri}, for which $\omega(I_{r1}) = \omega(h_{r1}) = \Omega_1$, $\omega(I_{r2}) = \omega(h_{r2}) = \Omega_2$, with $|I_{r1} - I_{r2}| \sim |h_{r1} - h_{r2}| \gg \sqrt{\varepsilon}$. For instance, if one of the amplitudes f_1, f_2 is equal to zero, (7.4) becomes integrable close to I_{r1} or I_{r2} since the Hamiltonian includes only one resonant term. In that case, the motion is stable and we can reduce it to the resonant Hamiltonian H_r,

$$H_r(p, \psi) = \frac{p^2}{2M} + \varepsilon V \cos\psi, \tag{7.5}$$

where

$$V = \frac{a(I_{ri}) f_i}{2}, \quad p = I - I_{ri}\ (m, n = 1), \quad \psi = \vartheta - \tau_i, \quad M^{-1} = \omega'(I_{ri}),$$

and thus (see Section 2.3 for details of the equations)

$$M^{-1} = \frac{\mathrm{d}\omega}{\mathrm{d}I}|_{I_{ri}} = \left(\frac{\mathrm{d}\omega}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}I}\right)_{I_{ri}} = \left(\frac{\mathrm{d}\omega}{\mathrm{d}a}\right)_{I_{ri}} \left(\frac{\mathrm{d}I}{\mathrm{d}a}\right)^{-1}_{I_{ri}} = \frac{\beta^2}{a^2(I_{ri})}.$$

Recalling that the frequency width is (5.20)

$$(\Delta\omega)_r = 2\sqrt{\frac{\varepsilon V_{mn}(I_{ri})}{M}} = \beta\sqrt{\frac{2\varepsilon f_i}{a}}.$$

Assume that both resonant domains are "far enough", this means that

$$|\Omega_1 - \Omega_2| = |\omega_{r1} - \omega_{r2}| \gg \sqrt{\varepsilon},$$

then it is plausible to assume that due to the smallness of the perturbation the motion would be confined to the domain of each resonance, depending on the initial conditions (h). Indeed, since

$$\omega = \sqrt{2}\beta h^{1/4},$$

then from the resonance condition

$$\omega = \Omega_i \quad \to \quad h_{ri} = \left(\frac{\Omega_i}{\sqrt{2}\beta}\right)^4$$

and thus, if for instance $\Omega_1 < \Omega_2$ then ($\beta \sim 1$)

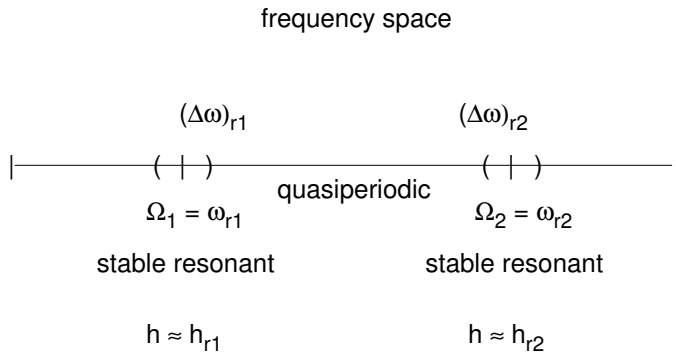

Figure 7.4: Frequency space for the situation in which $|\Omega_1 - \Omega_2| \gg \sqrt{\varepsilon}$ and $\Omega_1 < \Omega_2$. The stable resonant motion will be confined either to a neighborhood of ω_{r1} or ω_{r2} depending on the energy level h. For $|h - h_{ri}| \gg \sqrt{\varepsilon}$ the motion is non-resonant or quasiperiodic.

$$h_{r1} \approx \frac{\Omega_1^4}{4}, \qquad h_{r2} \approx \frac{\Omega_2^4}{4}$$

so $h_{r1} \ll h_{r2}$. This situation is schematically represented in Fig. 7.4. For instance, if $h \approx h_{r1}$ the motion will be resonant around this energy value and the term in the perturbation involving $\cos(\vartheta - \tau_2)$ becomes non-resonant and could be averaged out. Clearly the opposite situation occurs when $h \approx h_{r2}$.

The scenario is quite different when $|\Omega_1 - \Omega_2| \sim \sqrt{\varepsilon}$, that is when both resonance domains are close enough. Therefore the motion could not be confined to any of these domains and the motion could jump from one resonance to the other. As we have already mentioned, since $\Omega_1 \sim \Omega_2$ this new domain of chaotic motion is small. The limit case when the resonances overlap is then (see Fig. 7.5)

$$\omega_{r1} + (\Delta\omega)_{r1} = \omega_{r2} - (\Delta\omega)_{r2} \quad \rightarrow \quad (\Delta\omega)_{r1} + (\Delta\omega)_{r2} = \omega_{r2} - \omega_{r1} = \Omega_2 - \Omega_1.$$

Since $(\Delta\omega)_{ri} = \beta\sqrt{2\varepsilon f_i / a(I_{ri})}$, then the overlap criterion provides the (theoretical) critical value ε_c :

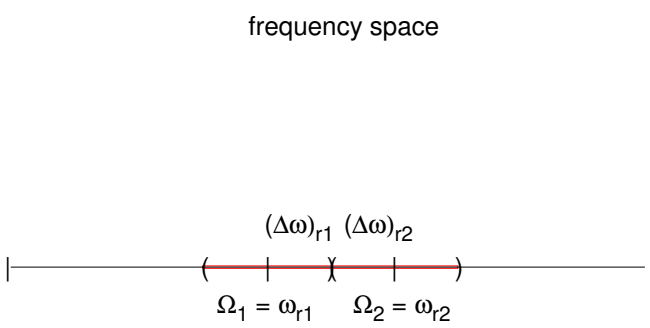

Figure 7.5: Frequency space in case of $|\Omega_1 - \Omega_2| \lesssim \sqrt{\varepsilon}$. The resonance domains "touch" each other and therefore they overlap. The resulting (chaotic) domain for the motion is indicated in red.

$$\beta\sqrt{2\varepsilon_{\rm c}}\left(\sqrt{\frac{f_1}{a(I_{r1})}}+\sqrt{\frac{f_2}{a(I_{r2})}}\right)=\Omega_2-\Omega_1\equiv\Delta\Omega.$$

Take for instance $f_1 = f_2 = f$ and since $\Omega_1 \approx \Omega_2 \to \omega_1 \approx \omega_2 \to I_{r1} \approx I_{r2} = I_r$, then the above condition for $\varepsilon_{\rm c}$ reduces to

$$2\beta\sqrt{\frac{2\varepsilon_{\rm c} f}{a(I_r)}}=\Delta\Omega \quad \to \quad \varepsilon_{\rm c}=\frac{a(I_r)\Delta\Omega^2}{8f\beta^2},$$

and since $\omega = \beta a$ and $\omega \approx \omega_{ri} \approx \Omega_i$ we finally get for the theoretical critical size of the perturbation

$$\varepsilon_{\rm c}\approx\frac{\omega\Delta\Omega^2}{8f\beta^3}.$$

Then if $\varepsilon \geq \varepsilon_{\rm c}$ the motion will not be confined to the domain of a given resonance and an orbit could visit the whole overlapped domain. However, as we have already said, the latter is generally small since we have only considered two resonances. In the next Chapter we will consider the case when many resonance are involved.

Chapter 8

Resonance interaction II

As mentioned, in this chapter we discuss qualitatively the motion under the influence of many resonances. A very convenient way to do this is to consider a system with discrete time described by finite difference equations. This kind of representation of a dynamical system is called a *mapping* or simply a *map*.

8.1 The standard map

Let us consider, for instance, a 2D map

$$(I, \vartheta) \to (I', \vartheta'), \qquad I, I' \in G \subset \mathbb{R}, \qquad \vartheta, \vartheta' \in S^1,$$

defined by

$$\begin{aligned} I' &= I + Kf(\vartheta), \\ \vartheta' &= \vartheta + I' = \vartheta + I + Kf(\vartheta), \end{aligned} \tag{8.1}$$

where K is a constant parameter and $f(\vartheta)$ is some well behaved periodic function.

In general, a map describes the evolution of a dynamical system providing the values of the phase space variables at multiples of a given time. Indeed, let T be this time, then if $I = I(t)$, $\vartheta = \vartheta(t)$, then $I' = I(t+T), \vartheta' = \vartheta(t+T)$.

An alternative notation then could be

$$\begin{aligned} I_{n+1} &= I_n + Kf(\vartheta_n), \\ \vartheta_{n+1} &= \vartheta_n + I_{n+1}, \end{aligned}$$

where $I_n = I(t_0 + nT), \vartheta_n = \vartheta(t_0 + nT)$, $n = 0, 1, \ldots, N \to \infty$ and t_0 an arbitrary initial time that we set $t_0 = 0$ in what follows. Anyway, we will keep the notation introduced in (8.1). Fig. 8.1 schematically represents the values of the action provided by the map and the continuous evolution of the action as the solution of its given differential equation (see below).

The map (8.1) thus defined could be thought as a sequence of canonical transformations, with the generating function

$$F(I', \vartheta) = I'\vartheta + \frac{1}{2}I'^2 + KV(\vartheta), \qquad -\frac{\mathrm{d}V}{\mathrm{d}\vartheta}(\vartheta) = f(\vartheta).$$

Recalling that

$$\begin{aligned} I &= \frac{\partial F}{\partial \vartheta} = I' + K\frac{\mathrm{d}V}{\mathrm{d}\vartheta}(\vartheta), \\ \vartheta' &= \frac{\partial F}{\partial I'} = \vartheta + I', \end{aligned}$$

we reobtain the map. Therefore it preserves the volume (area in this case) of phase space. Two dimensional canonical maps are usually called *area preserving maps* and as we shall see, they represent the motion in some low dimensional Hamiltonian system.

Let us note that (8.1) for $K = 0$ represents the motion on a 1D torus in an integrable system. From the equation for the action we get $I(t_0 + nT) = I(t_0) = I$ and thus I is constant. From the second equation we get

$$\begin{aligned} \vartheta(t_0 + T) &= \vartheta(t_0) + I(t_0 + T) = \vartheta(t_0) + I, \\ \vartheta(t_0 + 2T) &= \vartheta(t_0 + T) + I(t_0 + 2T) = \vartheta(t_0) + 2I \\ &\cdots\cdots \\ \vartheta(t_0 + nT) &= \vartheta(t_0 + (n-1)T) + I(t_0 + nT) = \vartheta(t_0) + nI. \end{aligned}$$

Therefore we get $I = \text{const}$ and $\vartheta(t) = nI + \vartheta(t_0)$ which are the equations of motion (for the integer time n) on a given torus with frequency $\omega(I) = I$. Clearly this is the corresponding equation for the angle in the 1D (integrable) Hamiltonian $H_0 = I^2/2$.

Let us try now to derive the differential equations or Hamiltonian that lead to the above finite difference equations or map (8.1) for an arbitrary value of K. Certainly, if we find explicit expressions like $\dot{I} = u$ and $\dot{\vartheta} = w$ then the map should be thought as an integration of the latter differential

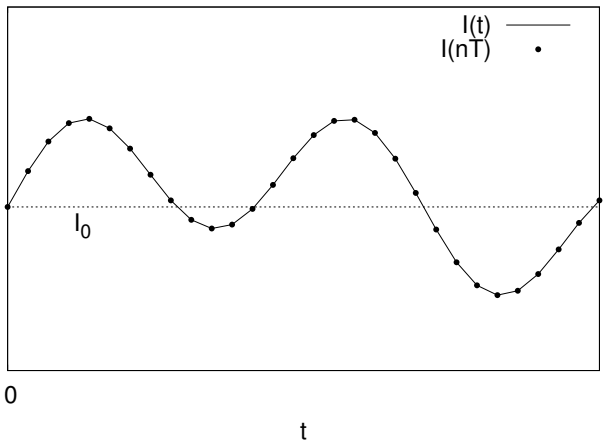

Figure 8.1: Schematic representation of the values provided by the map for the action (dots) and the continuous time evolution of I (solid line). The doted line represents the time evolution of the action in case of $K = 0$.

equations over a time interval $\Delta t = T$ such that, for instance $\int_t^{t+T} \dot{I} \mathrm{d}t = I(t+T) - I(t) \equiv I' - I = U(t+T) - U(t)$, where U is the primitive of u evaluated at the values of the phase space variables at $t+T$ and t respectively. Fig. 8.1 represents the continuous and discrete time evolution of I as a solution of the differential equation and as the iteration of the map, respectively. Clearly in a similar fashion we get for the phase $\vartheta' - \vartheta = W(t+T) - W(t)$.

At first sight we could approximate for K and T small

$$\frac{I' - I}{T} = \frac{K}{T} f(\vartheta) \quad \rightarrow \quad \dot{I} \approx \frac{K}{T} f(\vartheta),$$
$$\frac{\vartheta' - \vartheta}{T} = \frac{I'}{T} \quad \rightarrow \quad \dot{\vartheta} \approx \frac{I}{T},$$

arriving then to an autonomous 1D system. However let us proceed in a more "rigorous" way.

Let us take a time interval (t_1, t_2) such that $|t_2 - t_1| < T$ and assume that after the $(n-1)$-th iteration (that is after a motion time $(n-1)T$), it happens that $nT \in (t_1, t_2)$ as Fig. 8.2 illustrates. Thus if

$$t_1 < nT < t_2, \quad \rightarrow \quad I(t_2) = I(t_1) + K f(\vartheta_1).$$

And it is easy to see that if

$$t_1 < t_2 < nT, \quad \rightarrow \quad I(t_2) = I(t_1).$$

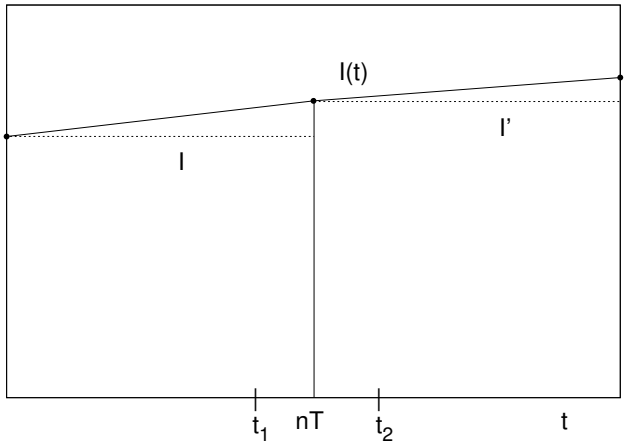

Figure 8.2: Continuous time evolution of $I(t)$ and the discrete one given by the map. In this figure it is assumed that $nT \in (t_1, t_2)$ so that the jump in I occurs within this time interval (see text).

Therefore we can put the above conditions as

$$\begin{aligned} \dot{I}(t) &= 0 \qquad \text{if} \quad nT \notin (t_1, t_2), \\ \int_{t_1}^{t_2} \dot{I}(t) &= Kf(\vartheta_1) \qquad \text{if} \quad nT \in (t_1, t_2). \end{aligned}$$

Thus from the above relations, for $t_2 \to t_1$ and since t_1 is an arbitrary point, we can write

$$\dot{I}(t) = Kf(\vartheta(t))\delta(t - nT), \tag{8.2}$$

where δ is the δ-function. Now since T is the period of iteration of the map, we can introduce the frequency $\Omega = 2\pi/T$ and the phase $\tau = \Omega t + \tau_0, \tau \in S^1$, with τ_0 a constant. Therefore we can take τ as a time-like variable instead of t, and recalling that the δ-function is a distribution

$$\delta(t - nT) = \delta(\tau - 2\pi n)\frac{\mathrm{d}\tau}{\mathrm{d}t} = \Omega\delta(\tau),$$

therefore (8.2) becomes

$$\dot{I} = K\Omega f(\vartheta)\delta(\tau), \tag{8.3}$$

which is only valid for one period of iteration of the map (for instance as it is shown in Fig. 8.2) or just on a given time interval. Since our aim is to

find the differential equation for any time and due to the periodicity of the iterations (T in t or 2π in τ) we write (8.3) as

$$\dot{I} = K\Omega f(\vartheta)\delta_{2\pi}(\tau), \tag{8.4}$$

where $\delta_{2\pi}(\tau)$ is the 2π periodic δ-function defined through its Fourier expansion coefficients,

$$\begin{aligned} a_0 &= \frac{2}{2\pi}\int_{-\pi}^{\pi}\delta_{2\pi}(\tau)\mathrm{d}\tau = \frac{1}{\pi}, \\ a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi}\delta_{2\pi}(\tau)\cos(n\tau)\mathrm{d}\tau = \frac{1}{\pi}, \\ b_n &= 0. \end{aligned}$$

Therefore

$$\delta_{2\pi}(\tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\tau) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n\tau). \tag{8.5}$$

Let us now look for the differential equation for ϑ. We have already found that if $nT \notin (t_1, t_2)$ then $I(t_2) = I(t_1)$ and thus

$$\vartheta(t_2) = \vartheta(t_1) + I(t_1).$$

Introducing an integer partition of (t_1, t_2)

$$\alpha = \{[t_1] = k_1 < k_2 < \cdots < k_m = [t_2]\},$$

in any cell (k_i, k_{i+1}) the action takes the very same value $I(k_i) = I(t_1)$, so when $m \to \infty$

$$\vartheta(t_2) = \vartheta(t_1) + \sum_{k_i \in \alpha} I(k_i) = \vartheta(t_1) + I(t_1)(t_2 - t_1),$$

where t_1, t_2 are adimensional times. Introducing then the dimensional time $s = tT$ the above equation reads

$$\vartheta(s_2) = \vartheta(s_1) + \frac{I(s_1)}{T}(s_2 - s_1) \quad \to \quad \frac{\vartheta(s_2) - \vartheta(s_1)}{s_2 - s_1} = \frac{I(s_1)}{T},$$

then taking $\lim_{s_2 \to s_1}$ and since s_1 is arbitrary we get

$$\dot{\vartheta} = \frac{I}{T}. \tag{8.6}$$

Now let us consider the opposite case, when $nT \in (t_1, t_2)$. Thus for $t \in (t_1, t_2)$ it is

$$I(t) = I(t_1) \qquad \text{if} \qquad t_1 < t < nT < t_2,$$
$$I(t) = I(t_2) \qquad \text{if} \qquad t_1 < nT < t < t_2.$$

By means of the very same arguments considered above we write

$$\vartheta(t_2) = \vartheta(t_1) + I(t_1)(nT - t_1) + I(t_2)(t_2 - nT),$$

or denoting $\Delta\vartheta = \vartheta(t_2) - \vartheta(t_1)$,

$$\Delta\vartheta = I(t_1)(nT - t_1) + I(t_2)(t_2 - nT).$$

Setting $t_1 = nT - \epsilon/2$, $t_2 = nT + \epsilon/2$ with $\epsilon > 0$, then $\Delta t = \epsilon$ leading to

$$\Delta\vartheta = \frac{I(nT - \epsilon/2) + I(nT + \epsilon/2)}{2}\Delta t,$$

and using again the dimensional time $s = tT,\ \Delta s = T\Delta t$, we get

$$\frac{\Delta\vartheta}{\Delta s} = \frac{I(nT - \epsilon/2) + I(nT + \epsilon/2)}{2T}.$$

Then when $\epsilon \to 0^+$ we get

$$\dot{\vartheta} = \frac{I(nT^-) + I(nT^+)}{2T},$$

where $t = nT$ is the point of discontinuity of $I(t)$. Recalling that $\dot{I}$ is given by a Fourier expansion, it does $I(t)$ and, from the Dirichlet condition, at the points of discontinuity the Fourier series converges to the semi-sum of the function at that point. Thus we finally obtain

$$\dot{\vartheta} = \frac{I}{T},$$

which is the very same expression than (8.6).

Introducing the non-canonical change of variable $J = I/T$,[1] from (8.4)-(8.6), the differential equations become ($T = 2\pi/\Omega$)

$$\begin{aligned} \dot{J} &= \frac{K}{2\pi}\Omega^2 f(\vartheta)\delta_{2\pi}(\tau), \\ \dot{\vartheta} &= J. \end{aligned} \tag{8.7}$$

We observe then from (8.7) that the exact differential equations for I (or J) and ϑ differ from the approximate ones since the former correspond to a non-autonomous system or a 1.5 degree of freedom Hamiltonian. Indeed, equations (8.7) derive from the Hamiltonian

$$H(J,\vartheta,\tau) = \frac{J^2}{2} + \frac{K}{2\pi}\Omega^2 V(\vartheta)\delta_{2\pi}(\tau), \qquad -\frac{\mathrm{d}V}{\mathrm{d}\vartheta} = f(\vartheta), \qquad J = \frac{I}{T}; \tag{8.8}$$

since

$$\dot{J} = -\frac{\partial H}{\partial \vartheta}, \qquad \dot{\vartheta} = \frac{\partial H}{\partial J},$$

lead to the exact differential equations for J and ϑ.

The equations of motion (8.7) are completely equivalent to the original map (8.1) and thus we note that the map describes the evolution of $I(t), \vartheta(t)$ under the effect of a periodic perturbation of period T. The latter being a sequence of small kicks and should lead to an infinite set of resonances. Indeed, let us see this with a particular example of $f(\vartheta)$.

Let $f(\vartheta) = \sin\vartheta, \Omega = 1$ ($T = 2\pi$), then $\tau = t + t_0$ and without loss of generality take $t_0 = 0$. This selection of $f(\vartheta)$ leads to $V(\vartheta) = \cos\vartheta$. Therefore we arrive to the so-called *standard map* (SM)

$$\begin{aligned} I' &= I + K\sin\vartheta, \\ \vartheta' &= \vartheta + I', \end{aligned} \tag{8.9}$$

[1]This change of variables though preserve the Hamilton equations is not canonical since the Jacobian $\partial(I,\vartheta)/\partial(J,\vartheta) = T^{-1}$ and thus the area is not preserved, it shrinks by a factor $1/T$, but this does not impose any limitation to the introduction of the variable J.

whose corresponding Hamiltonian, with the help of (8.5) and (8.8), takes the form

$$H(J,\vartheta,\tau)=\frac{J^2}{2}+\frac{K}{2\pi}\cos\vartheta\left\{\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos(nt)\right\}. \tag{8.10}$$

Using again the trigonometric identity

$$\cos a\cos b=\frac{\cos(a+b)+\cos(a-b)}{2},$$

the Hamiltonian (8.10) can be written as

$$H(J,\vartheta,\tau)=\frac{J^2}{2}+\frac{K}{4\pi^2}\cos\vartheta+\frac{K}{4\pi^2}\sum_{n=1}^{\infty}[\cos(\vartheta+nt)+\cos(\vartheta-nt)]. \tag{8.11}$$

Introducing the parameter $k=K/4\pi^2$ and letting $n\in\mathbb{Z}/0$ we finally obtain

$$H(J,\vartheta,\tau)=\frac{J^2}{2}+k\cos\vartheta+k\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\cos(\vartheta-nt). \tag{8.12}$$

Therefore the Hamiltonian that describes the SM is, for k relatively small, a pendulum plus a periodic time dependent perturbation. Note that all the Fourier harmonics have the same coefficient, k, which in fact is identical to the one corresponding to the pendulum model.

Let us take as the unperturbed Hamiltonian $H_0=J^2/2$, then the unperturbed frequency is $\dot\vartheta=\omega=\partial H_0/\partial J=J$. Then from (8.12), for k not too large, the resonances of the systems are

$$\dot\vartheta_r-n=0,\qquad J_r=n,\qquad n\in\mathbb{Z}, \tag{8.13}$$

where we have included $n=0$ that corresponds to the resonant term $\cos\vartheta$. Therefore we see that the full set of first order resonances is

$$\mathcal{R}_J=\{J_r: J_r^{(n)}=n,\, n\in\mathbb{Z}\}\quad\text{or}\quad \mathcal{R}_I=\{I_r: I_r^{(n)}=2\pi n,\, n\in\mathbb{Z}\}.$$

All first order resonances have the very same amplitude so their half width is identical in both frequency and action space ($\omega=J$), being the latter

$$(\Delta J)_r^{(n)}=(\Delta\omega)_r^{(n)}=2\sqrt{k}, \tag{8.14}$$

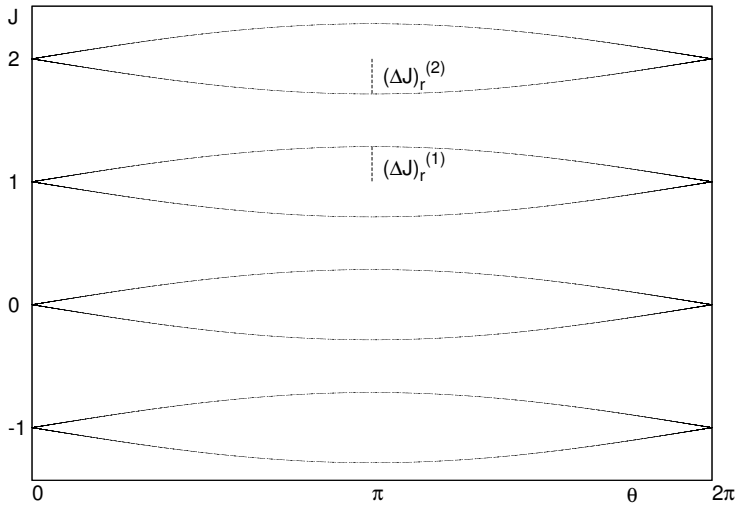

Figure 8.3: Schematic representation of the integer resonances in the SM considering only $n = -1, 0, 1, 2$. Each resonance has a half width $(\Delta J)_r^{(i)} = 2\sqrt{k}$, which is the same for all resonances. In this figure it is assumed that no overlap of integer resonances occurs.

since $M = 1, \varepsilon V_{mn} = k$.

The above set of resonances can be also derived from the map. Indeed, a resonance should be thought in the map as when the phase return to the very same point after one iteration. If we impose this condition also for the action, we get the fixed points of the map. Recalling that the SM (8.9) is also invariant under the translation $I \to I \pm 2\pi$, then from the second line in (8.9)[2] setting $I' = I$ we obtain

$$I_r = 0 \qquad \mod 2\pi,$$

and thus the resonance set is the same as that derived above: $I_r^{(n)} = 2\pi n$, $n \in \mathbb{Z}$. Thus any integer resonance could be modelled by the same pendulum Hamiltonian since the map is invariant under the shift $I \to I - I_r$ (or $J \to J - J_r$).

Fig. 8.3 represent this set of resonances and their corresponding half widths. Note that if we let $J, I \in \mathbb{R}$ the phase space is a cylinder and

[2]The first equation in (8.9) tells us then the location of the fixed point in the coordinate ϑ, in the present case, $K \sin \vartheta = 0$ implies $\vartheta = 0, \pi, \mod 2\pi$.

the resonances range is $J_r^{(n)} = -\infty, \ldots, 0, \ldots, \infty$, while if $J, I \in S^1$, then for $J \in (0,1)$ or $I \in (0, 2\pi)$ only one resonance is present $J = 0^+, 1^-$ (see below), since the "sides" $J = 0, 1$ $(I = 0, 2\pi)$ should be identified.

From Fig. 8.3 we can easily see that the condition for the overlap of first order or integer resonances is

$$(\Delta J)_r^{(n)} + (\Delta J)_r^{(n-1)} = 1, \quad \rightarrow \quad 4\sqrt{k} = 1, \quad \rightarrow \quad k_T = \frac{1}{16},$$

and since $k = K/4\pi^2$,

$$K_T = \frac{\pi^2}{4} \approx 2.5,$$

where the subscript T refers to the theoretical estimation of the critical value of the perturbation parameter, above which the stochastic instability arises. As we shall see this value for K_T largely overestimates the real critical one, since numerical experiments reveal an empirical value of $K_E \approx 1$.

Anyway, since in the SM all resonances are identical, if two of them overlap ($K > K_E$), then the full set of resonances $\mathcal{R}_J$ does, as it is evident from Fig. 8.3. Therefore, if we consider $J \in \mathbb{R}$ then we should expect a large variation of the unperturbed action, since it could range, for instance from $J \approx 0$ to $J \approx n \gg 1$. On the other hand for $K < K_E$, the perturbation is small enough such that first order resonances do not overlap and therefore the variation of J is just confined to the width of a given resonance, $|\Delta J| \lesssim 2\sqrt{k} \ll 1$.

The gross instability due to the overlap of first order resonances is caused by to heteroclinic intersections between the stable and unstable manifolds of two nearby resonances. As we have already mentioned, the resulting motion looks like "random" as if the system were dominated by a stochastic interaction.

In order to observe this behavior, let us proceed with numerical experiments. Let us consider the SM (8.9) and take $J \in S^1$ such that only one first order integer resonance is present, $J = 0^+$ and $J = 1^-$. We iterate the map up to $N = 2000$ for 300 initial conditions very close to the J-axis ($\vartheta \approx 0$) and for four different values of K, such that $K < K_E$ and $K > K_E$.

Fig. 8.4 shows the (J, ϑ) space for $K = 0.2, 0.7, 1.0$ and 2.5. From the figure for $K = 0.2$ we clearly see the resonance corresponding to $n = 0$, that shows up around $J = 0$ and $J = 1$, since this two sides of the unit square should be identified. The resonance in fact look like a pendulum model where oscillations are confined to a small domain of size $2\sqrt{k} = 2\sqrt{K/4\pi^2} =$

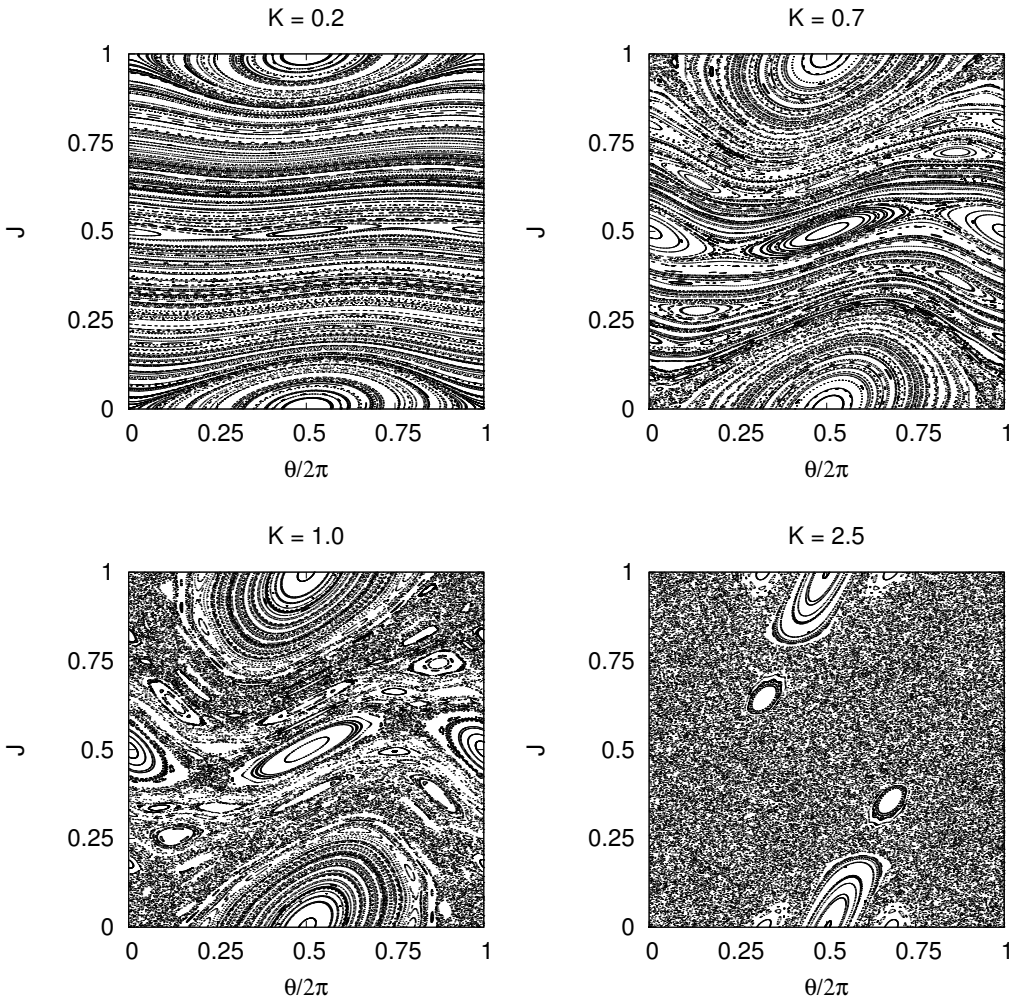

Figure 8.4: Phase space of SM for $K < K_E$ and $K > K_E$ restricted to $J \in S^1$ after 2000 iteration of 300 initial conditions taken almost along the J-axis. The single integer resonance appears around $J = 0, 1$ where these two sides of the unit square should be identified (see text).

$\sqrt{K}/\pi \approx 0.14$ for $K = 0.2$ around $J = 0, 1$, in good agreement with experimental value. Therefore a given orbit with initial conditions $J \approx 0, 1$ and $0 < \vartheta < 2\pi$ will describe a smooth curve within the oscillation regime. For initial conditions such that $J \gtrsim 2\sqrt{k}$ the orbit is no longer trapped in the resonance and it moves also in a smooth curve but in the rotation regime,

like the numerical experiments show for $K = 0.2$.

For $K = 0.7$ the integer resonance looks distorted and the separatrix shows as a narrow layer where the motion resembles "scattered points". Indeed, being the separatrix of the pendulum a rather unstable trajectory, it is expected that the motion close to the separatrix would be largely affected. More precisely, all the invariant curves (tori) close to this asymptotic orbit are destroyed by the perturbation. The motion does not proceed any longer over a given torus (smooth curve in this case); it does not exist any local integral near the separatrix and the dynamics in this thin region of the phase plane seems not well represented by a pendulum Hamiltonian for J as it is, for instance, close to the center of the resonance. Therefore due to the perturbation we say that the unperturbed separatrix becomes a thin layer of stochastic or chaotic motion of finite width (in next chapter we shall discuss this effect in detail).

Besides, for $K = 0.7$ we observe at $J \approx 0.5$ two "islands" or pendulum models, but of smaller size than for $J \approx 0(1)$. In fact these islands at $J \approx 0.5$ are also present (even smaller) in the plot for $K = 0.2$. Moreover, another set of distinguishable islands appear at $J \approx 1/3, 2/3$ as well as many smaller ones (see next section). Most of all these chains of islands seem to be also confined by a thiner layer of chaotic motion.

When K increases, we note, for instance for $K = 1$, that both sides of the integer resonances seem to be connected, no significant barriers to the chaotic motion are present, so J could exhibit a variation $\Delta J \approx 1$, which is large in comparison with the case of $K = 0.7$, where at most $\Delta J \approx \sqrt{K}/\pi \approx 0.27$ within the integer resonance. Indeed most tori are destroyed due to the overlap of resonances and the motion is no longer confined to the domain of one resonance (except close to the center of any of them). This is much more evident for $K = 2.5$ where the motion is stable just within small domains close to the centers of few islands.

In Fig. 8.4 only one integer resonance is included. If, for instance we would like to consider q of them a simple modification of the map (8.9) needs to be introduced. It is also simpler to take both, the action and the angle in the unit interval $(0, 1)$, defining new variables (x, p) such that

$$\vartheta = 2\pi x, \qquad I = 2\pi q p, \quad q \in \mathbb{N};$$

which after substituting in the SM (8.9) leads to

$$p' = p + \frac{K}{2\pi q} \sin 2\pi x, \qquad x' = x + q p', \qquad p, x \in S^1.$$

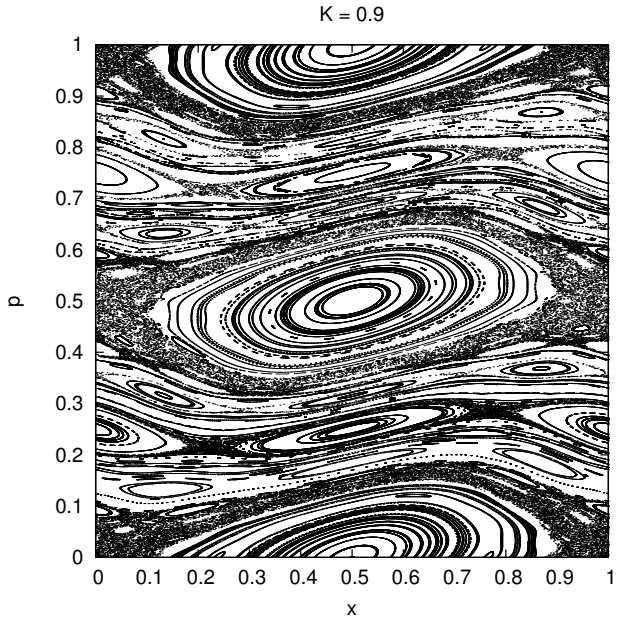

Figure 8.5: Phase space of the SM for $q = 2$ using p, x mod 1 as variables for $K = 0.9$, a large set of initial conditions and $N \gg 1$ iterates (higher resolution than Fig. 8.4).

To see that q integer resonances are present, note that $\Delta I = 2\pi q \Delta p$, and since resonances are separated $\Delta I_r = 2\pi$, then $\Delta p_r = 1/q$. The resonances appear then at $p_r^{(n)} = n/q$. For example, taking $q = 2$ we get twice the picture given by Fig. 8.4 as it is shown in Fig. 8.5.

8.2 High-order resonances

We have already found that the SM has an associated Hamiltonian (8.12) which can be rewritten as

$$H(J, \vartheta, \tau) = \frac{J^2}{2} + k \sum_{n=-\infty}^{\infty} \cos(\vartheta - nt), \qquad J = \frac{I}{2\pi}, \quad k = \frac{K}{4\pi^2}. \tag{8.15}$$

Assuming again k small, $H_0(J) = J^2/2$ is the unperturbed Hamiltonian and introducing a canonical transformation $(J, \vartheta) \to (J_1, \vartheta_1)$ in order to kill all

terms of $\mathcal{O}(k)$ in the perturbation, the latter becomes of $\mathcal{O}(k^2)$. Accordingly to results of previous chapters, this transformation could only exist "far away" from the first order resonances defined in (8.13): $J_r = n$, $n \in \mathbb{Z}$. Let $F(J_1, \vartheta)$ be the generating function defined in the usual way

$$F(J_1, \vartheta, t) = J_1\vartheta + k\Phi(J_1, \vartheta, t),$$

where Φ should be determined. The corresponding transformation equation are then

$$J = \frac{\partial F}{\partial \vartheta} = J_1 + k\Phi_\vartheta, \quad \vartheta_1 = \frac{\partial F}{\partial J_1} = \vartheta + k\Phi_{J_1}, \quad H_1 = H + k\frac{\partial F}{\partial t} = H + \Phi_t.$$

Replacing J in (8.15), H_1 reads

$$H_1 = \frac{J_1^2}{2} + kJ_1\Phi_\vartheta + \frac{1}{2}k^2\Phi_\vartheta^2 + k\Phi_t + k\sum_{n=-\infty}^{\infty}\cos(\vartheta - nt),$$

or in terms order in k,

$$H_1 = \frac{J_1^2}{2} + k\left(J_1\Phi_\vartheta + \Phi_t + \sum_{n=-\infty}^{\infty}\cos(\vartheta - nt)\right) + \mathcal{O}(k^2),$$

where the term at $\mathcal{O}(k^2)$ is $k^2\Phi_\vartheta^2/2$. Thus to kill all terms of order k in the perturbation, the expression between brackets should vanish,

$$J_1\Phi_\vartheta + \Phi_t + \sum_{n=-\infty}^{\infty}\cos(\vartheta - nt) = 0. \tag{8.16}$$

.

Let take

$$\Phi(J_1, \vartheta, t) = \sum_{n=-\infty}^{\infty} a_n(J_1)\sin(\vartheta - nt),$$

then

$$\Phi_\vartheta = \sum_{n=-\infty}^{\infty} a_n(J_1)\cos(\vartheta - nt), \quad \Phi_t = -\sum_{n=-\infty}^{\infty} na_n(J_1)\cos(\vartheta - nt).$$

Replacing in (8.16) we get

$$\sum_{n=-\infty}^{\infty} \{(J_1 - n)a_n(J_1) + 1\} \cos(\vartheta - nt) = 0,$$

and therefore

$$a_n(J_1) = \frac{1}{n - J_1}, \qquad J_1 \neq n \in \mathbb{Z}.$$

Then Φ takes the final form

$$\Phi(J_1, \vartheta, t) = \sum_{n=-\infty}^{\infty} \frac{1}{n - J_1} \sin(\vartheta - nt), \tag{8.17}$$

thus, $H_1 = J_1^2/2 + k^2\Phi_\vartheta^2/2$ is

$$H_1 = \frac{J_1^2}{2} + \frac{k^2}{2} \sum_{m,n=-\infty}^{\infty} \frac{1}{(n - J_1)(m - J_1)} \cos(\vartheta - nt) \cos(\vartheta - mt), \tag{8.18}$$

only valid for $J_1 \neq m, n \in \mathbb{Z}$. Using the trigonometric identity

$$\cos a \cos b = \frac{\cos(a + b) + \cos(a - b)}{2},$$

the sum (8.18) involves then the harmonics

$$\frac{1}{2} \cos(2\vartheta - (n + m)t) + \frac{1}{2} \underbrace{\cos((n - m)t)}_{\text{free oscillations}},$$

thus averaging out the oscillations, H_1 reduces to

$$H_1 = \frac{J_1^2}{2} + \frac{k^2}{4} \sum_{m,n=-\infty}^{\infty} \frac{1}{(n - J_1)(m - J_1)} \cos(2\vartheta - (n + m)t). \tag{8.19}$$

Since $\vartheta_1 = \vartheta + \mathcal{O}(k)$ in (8.19) replacing $\vartheta \to \vartheta_1$ the correction is then of $\mathcal{O}(k^3)$ that we can neglect in H_1. Therefore a new set of resonances appears at $\mathcal{O}(k^2)$,

$$2\dot{\vartheta}_1 - (n + m) = 0, \quad \to \quad J_{1r} = \frac{n + m}{2},$$

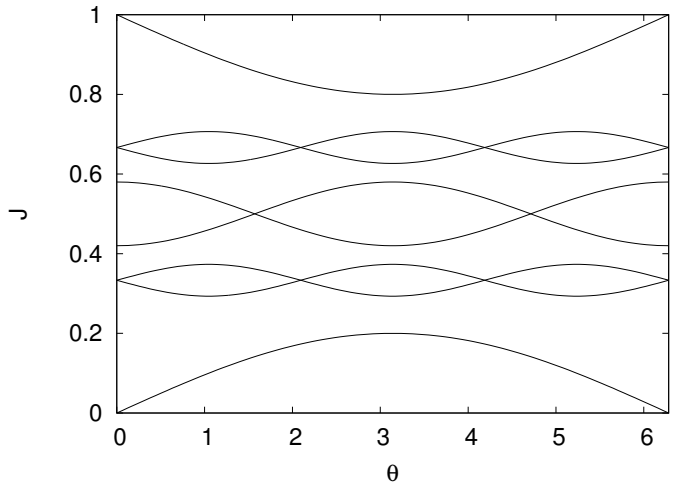

Figure 8.6: Representation of the location and widths of resonances in the SM up to $\mathcal{O}(K^3)$.

but the canonical transformation requires that J_1 not to be an integer as for instance (8.18) shows, so we write $m+n = 2p+1$, $p \in \mathbb{Z}$, and therefore we obtain the semi-integer set of resonances

$$J_{1r}^{(p)} = p + \frac{1}{2}, \qquad (\Delta J_1)_r \sim \mathcal{O}(K).$$

Therefore, since we assumed that $k \ll 1$ these resonances are smaller than the integer ones for which $(\Delta J)_r \sim \mathcal{O}(\sqrt{K})$. Thus for $J_1 \in (0,1)$ only one semi-integer resonance should appear $J_1 = 1/2$ of width $\mathcal{O}(K)$.

If we retain the neglected term of $\mathcal{O}(k^3)$ after the substitution $\vartheta \to \vartheta_1$ we would find a new set of resonances corresponding to harmonics of the form $\cos(3\vartheta_1 - lt)$, $l \in \mathbb{Z}$. Therefore in order to avoid small denominators $l = 3q+1$ and then these new set has the form

$$J_{1r}^{(q)} = q + \frac{1}{3}, \qquad (\Delta J_1)_r \sim \mathcal{O}(K^{3/2}).$$

Thus in the interval $(0,1)$ two new resonances appear, $J_1 = 1/3, 2/3$, of even smaller size than the semi-integer ones.

These new sets are clearly present in Fig. 8.4. Certainly we could go further to get resonances of even higher order in k with smaller size. Fig. 8.6

schematically represents, for $J \in (0,1)$ all resonances up to $\mathcal{O}(K^3)$ in the SM with their corresponding widths up to $\mathcal{O}(K^3)$. Now if we apply the overlap criterion but considering all resonances up to this order, it is possible to improve the value 2.5 for the critical K-value, obtaining $K_T \approx 1.35$.

From the discussions given in the last two chapters it turns clear that resonances interaction determines the dynamics of a non-linear system. One may think all resonances of a given system as many pendulum models, whose widths depend on the strength of the perturbation and any of them is "activated" after the selection of the initial conditions. For a very small perturbation, the size of all these pendula or resonances are negligible, even for the first order ones, the resonance interaction is so weak that in general the motion proceeds on slightly distorted unperturbed non-resonant tori. As the perturbation increases, the resonances become wider, the interaction between them begins to play a role, mostly around the separatrices of the pendula leading to narrow layers of chaotic motion. Therefore at this level of perturbation the picture corresponds to a large set of small pendula each of them surrounded by a thin chaotic layer, certainly with some other small chaotic regions due to the overlap of high-order resonances. For a perturbation strength larger than certain critical value, a major overlap of resonance takes place and the motion in phase space is mostly chaotic. Almost all pendulum models and invariant non-resonant tori are destroyed leading to a full connected region of strong chaotic motion. However in next section more arguments in this direction will be provided.

The above description is appropriate for low dimensional systems, like (8.15) or a system of two degrees of freedom. For multidimensional systems, the latter is a rough picture because as we discussed in the previous chapter, a stochastic instability may exist even for exponentially small perturbations.

Chapter 9

The Stochastic Layer

In previous chapters we have described in a qualitative way how resonance interaction for ε (or K) small, produces a modification of the separatrix of the resonances leading to a narrow layer of chaotic motion. Here we derive a quantitative estimate of the effect of such an interaction.

9.1 The whisker mapping

Let us consider the full Hamiltonian (6.1) in its real form

$$H(\boldsymbol{I},\boldsymbol{\vartheta}) = H_0(\boldsymbol{I}) + \varepsilon \sum_{\boldsymbol{m}\neq 0} V_{\boldsymbol{m}}(\boldsymbol{I}) \cos\left(\boldsymbol{m}\cdot\boldsymbol{\vartheta}\right), \qquad \varepsilon \ll 1. \tag{9.1}$$

Assume that only one term in (9.1) has $V_{\boldsymbol{m}} \neq 0$, so that the full Hamiltonian reduces to the resonant Hamiltonian H_r given in (6.12), take initial conditions such that $H_r \lesssim h_s = \varepsilon V_{\boldsymbol{m}}(\boldsymbol{I}_r)$, i.e. the system oscillates near the separatrix of the resonance. When switching on some other term $V_{\boldsymbol{m}'}$, slight (periodic) variations of H_r may cause the system to change drastically its motion, from oscillations to rotations and so on. Thus it is evident that the motion in the vicinity of the separatrix is extremely unstable.

Let us recall that measuring the distance from the separatrix by the relative energy $w = (H_r - h_s)/h_s$, we found that that the frequency, in a neighborhood of the separatrix, is

$$\omega(w) \approx \frac{\pi\omega_0}{\ln(32/|w|)} \ \to 0 \quad \text{as } \ w \to 0, \tag{9.2}$$

with $\omega_0 \propto \sqrt{\varepsilon}$ the small oscillation frequency already defined in (6.14), denoted by Ω in that case.

Following Chirikov's approach, let us see now how the motion looks near the separatrix when a perturbation is introduced. The simplest way to represent this is by means of a pendulum Hamiltonian acted upon by a periodic, time-dependent perturbation,

$$H(p,\psi,\tau) = H_r(p,\psi) + \mu V(\psi,\tau), \qquad \mu \ll 1, \tag{9.3}$$

where:

$$H_r = \frac{p^2}{2} - \omega_0^2 \cos\psi, \qquad \omega_0^2 = \varepsilon |V_{\boldsymbol{m}}| \ll 1, \tag{9.4}$$

is the resonant Hamiltonian (6.12) with $M = 1, p = p_1, \psi = \psi_1$, and

$$\mu V = \mu\omega_0^2 \cos\psi \cos\tau = \frac{\mu\omega_0^2}{2}[\cos(\psi-\tau) + \cos(\psi+\tau)]. \tag{9.5}$$

The perturbation V depends on the perturbing phase $\tau(t) = \Omega t + \tau_0 \in S^1$, $\Omega > 0$ being the perturbing frequency and $\tau_0 =$ it initial value. Let us take τ_0 as the value of τ when the pendulum crosses the equilibrium point, so τ_0 is the value of the phase of the perturbation each time ψ crosses $\psi = 0$.

Let us now compute the change in the unperturbed energy of the pendulum, $H_r \approx h_s = \omega_0^2(\varepsilon)$, over a half-period of oscillation or a period of rotation

$$T(w) = \frac{\pi}{\omega(w)} = \frac{1}{\omega_0} \ln\left(\frac{32}{|w|}\right).$$

To this end, we first calculate the time variation of H_r, to later integrate it over a whole period T. From the Hamiltonian (9.3)-(9.5), one readily finds

$$\begin{aligned} \dot{H}_r &= \frac{\partial H_r}{\partial t} + [H_r, H] = [H_r, H_r + \mu V] = \mu[H_r, V] = -\mu p(t)\frac{\partial V}{\partial \psi} \\ &= \frac{\mu\omega_0^2}{2} p(t) \left[\sin(\psi(t) - \tau) + \sin(\psi(t) + \tau)\right], \end{aligned} \tag{9.6}$$

where $[\,\cdot\,]$ denotes the Poisson bracket operator and the latter has to be integrated over $-T/2 \le t \le T/2$. Let us notice that, though there are explicit formulae for the unperturbed values of $p(t)$ and $\psi(t)$, they involve

elliptic functions or, eventually, Fourier series, so that such expressions are not convenient for our purpose. However, since the motion in the vicinity of the separatrix is nearly the same as that on the separatrix itself (except $T \to \infty$), we can write $p(t) \approx p_s(t)$, $\psi(t) \approx \psi_s(t)$ and $-\infty < t < \infty$; p_s and ψ_s given by (2.14) and (2.15):

$$p_s = \pm 2\omega_0 \cos\left(\frac{\psi_s}{2}\right), \qquad \psi_s(t) = 4\arctan\left(e^{\omega_0 t}\right) - \pi. \tag{9.7}$$

Therefore, taking one branch of the separatrix, for instance the plus sign in the first of (9.7) and assuming that the *slow phase* is $(\psi - \tau)$, we can average over the *fast phase* $(\psi + \tau)$ to obtain:

$$\Delta H_r \approx \mu\omega_0^3 \int_{-\infty}^{\infty} \mathrm{d}t \sin(\psi_s - \tau)\cos(\psi_s/2). \tag{9.8}$$

Using the trigonometric relation,

$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b)),$$

ΔH_r can be recast as

$$\Delta H_r \approx \frac{\mu\omega_0^3}{2} \int_{-\infty}^{\infty} \mathrm{d}t \left[\sin\left(\frac{3}{2}\psi_s - \tau\right) + \sin\left(\frac{1}{2}\psi_s - \tau\right)\right].$$

Further, replacing $\tau(t) = \Omega t + \tau_0$ and expanding

$$\sin(n\psi_s - \Omega t - \tau_0)) = \sin(n\psi_s - \Omega t)\cos\tau_0 - \cos(n\psi_s - \Omega t)\sin\tau_0,$$

for $n = 3/2, 1/2$ and taking into account that $\sin(n\psi_s - \Omega t)$ is odd and the odd term does not contribute to the integral, we obtain:

$$\Delta H_r \approx -\frac{\mu\omega_0^3}{2}\sin\tau_0 \left\{\int_{-\infty}^{\infty} \mathrm{d}t \cos\left(\frac{1}{2}\psi_s - \Omega t\right) + \int_{-\infty}^{\infty} \mathrm{d}t \cos\left(\frac{3}{2}\psi_s - \Omega t\right)\right\},$$

introducing

$$\lambda = \frac{\Omega}{\omega_0}, \qquad \hat{t} = \omega_0 t$$

the above expression can be rewritten as

$$\Delta H_r \approx -\frac{\mu\omega_0^2}{2}\sin\tau_0\left\{\int_{-\infty}^{\infty}\mathrm{d}\hat{t}\cos\left(\frac{1}{2}\psi_s-\lambda\hat{t}\right)+\int_{-\infty}^{\infty}\mathrm{d}\hat{t}\cos\left(\frac{3}{2}\psi_s-\lambda\hat{t}\right)\right\}.$$

Defining

$$\begin{aligned} A_m(\lambda) &= \int_{-\infty}^{\infty}\mathrm{d}s\cos\left(\frac{m}{2}\varphi(s)-\lambda s\right), \\ \varphi(s) &= 4\arctan e^s-\pi \end{aligned} \tag{9.9}$$

with m integer, the change in the unperturbed integral can be written as

$$\Delta H_r \approx -\frac{1}{2}\mu\omega_0^2\sin\tau_0\left[A_1\left(\frac{\Omega}{\omega_0}\right)+A_3\left(\frac{\Omega}{\omega_0}\right)\right], \tag{9.10}$$

The integral in (9.9) is known as the *Melnikov–Arnold Integral* (MAI, hereafter) and its evaluation may be found in the Appendix of Chirikov review (1979), however a detailed computation of the MAI is included in the last section of this chapter. Actually, $A_m(\lambda)$ is defined as the "mean value" of the improper integral (9.9), which in fact does not converge. A detailed computation of the MAI in the complex plane (see next section), shows that the integral

$$A_m(\lambda,\hat{t})=\int_{-\hat{t}}^{\hat{t}}\mathrm{d}s\cos\left(\frac{m}{2}\left(4\arctan e^s-\pi\right)-\lambda s\right)$$

oscillates with an amplitude of the order of $1/\lambda\sim\sqrt{\varepsilon}$ for $\hat{t}\gg 1$ and, therefore, the limit for $\hat{t}\to\infty$ does not exist. But these periodic oscillations play no role in the problem of the stability of the motion so we can neglect them and retain only the aperiodic part. Being $\omega_0\sim\sqrt{\epsilon}$ small and $m\neq 0$, we can use the asymptotic value of the MAI for large positive $\lambda=\Omega/\omega_0$:

$$A_m(\lambda)\approx\frac{4\pi(2\lambda)^{m-1}}{(m-1)!}e^{-\pi\lambda/2},\qquad \lambda\gg m, \tag{9.11}$$

where the factorial should be replaced by the Gamma function, $\Gamma(m)$, for non–integer m.

Moreover, for negative λ it is

$$A_m(\lambda)=(-1)^m A_m(|\lambda|)e^{-\pi|\lambda|},\qquad \lambda<0, \tag{9.12}$$

which shows that, for large $|\lambda|$,

$$A_m(-|\lambda|) \ll A_m(|\lambda|).$$

The largest contribution to the MAI comes when $m\dot{\psi}_s\lambda > 0$. Indeed, in order to be

$$\phi \equiv \frac{m}{2}\psi_s(\hat{t}) - \lambda\hat{t}$$

a slow enough phase, it is required that $\dot{\phi} = m\dot{\psi}_s/2 - \lambda \approx 0$, and thus $m\dot{\psi}_s$ and λ should have the same sign.

Also a recurrence relationship for different m values can be derived, namely

$$A_{m+1} = \frac{2\lambda}{m}A_m - A_{m-1} \approx \frac{2\lambda}{m}A_m, \tag{9.13}$$

where the approximation holds for large λ.

Therefore, for the upper branch of the separatrix ($p_s > 0$) and using (9.10), (9.11) and (9.13), since

$$A_3(\lambda) + A_1(\lambda) = \lambda A_2(\lambda),$$

and

$$A_2(\lambda) \approx 8\pi\lambda e^{-\pi\lambda/2}, \qquad \lambda = \frac{\Omega}{\omega_0},$$

we obtain then that the variation of the unperturbed energy is,

$$\Delta H_r \approx -\frac{\mu\omega_0\Omega}{2}A_2\left(\frac{\Omega}{\omega_0}\right)\sin\tau_0 \approx -4\pi\mu\Omega^2 e^{-\frac{\pi\Omega}{2\omega_0}}\sin\tau_0. \tag{9.14}$$

For the lower branch of the separatrix ($p_s < 0$), since $\dot{\psi}_s < 0$, it is enough to take $\lambda = \Omega/\omega_0$ with opposite sign in the first equality in (9.14), so (9.12) shows that its contribution to ΔH_r is negligible (since $m\dot{\psi}_s\lambda < 0$). Thus, actually (9.14) is the total change of H_r for a complete period of oscillation, i.e. over $2T$. But, since the perturbation is symmetric, it depends on $(\psi \pm \tau)$ both with the same amplitude, see (9.5), then the upper and lower branches of the separatrix contribute in the same way to the MAI in each half period of oscillation. In other words, for $p_s > 0$ the fast angle is $(\psi + \tau)$ while, $(\psi - \tau)$ is the one for $p_s < 0$.

Following the above discussion, we readily conclude that (9.14) is the variation of H_r in each period of motion. This variation depends on τ_0, the

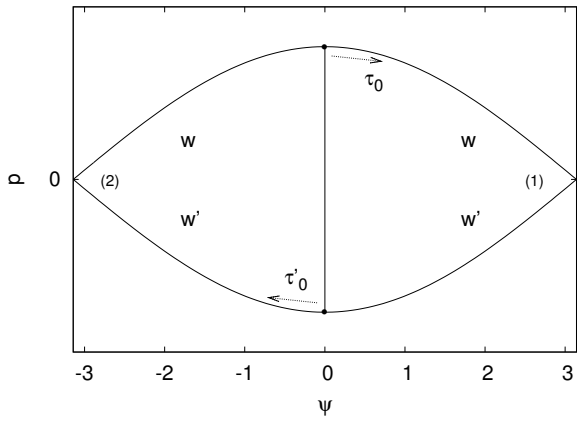

Figure 9.1: Representation of w, τ_0 evolution. The pendulum energy before crossing the point (1) is w and thus the perturbation phase has the value τ_0. After the pendulum reaches $(p, \psi) = (0, \pi)$ the energy changes to w' and after crossing the line $\psi = 0$ it does $\tau_0 \to \tau_0'$. The energy changes again at (2), when $(p, \psi) = (0, -\pi)$, so $w' \to w$ while the perturbing phase keeps the value τ_0' up to the next crossing with $\psi = 0$ when $\tau_0' \to \tau_0$.

value of the perturbing phase when $\psi = 0$. After a period of motion T, τ changes to

$$\tau' \equiv \tau(t + T) = \Omega t + \Omega \frac{\pi}{\omega} + \tau_0 \equiv \Omega t + \tau_0',$$

where,

$$\tau_0' = \tau_0 + \lambda \ln\left(\frac{32}{|w'|}\right); \qquad \lambda = \frac{\Omega}{\omega_0},$$

and

$$w' = \frac{(H_r' - \omega_0^2)}{\omega_0^2},$$

is the relative energy of the pendulum after crossing the surface $\psi = \pm\pi$, as Fig. 9.1 illustrates. Indeed, since the energy changes were computed along both branches of the separatrix and thus $\varphi(s) \to \pm\pi$ when $s \to \pm\infty$, while the phase changes at $\psi = 0$ (the map is said asynchronous).

Therefore, in the variables (w, τ_0), from (9.14)

$$H_r' = H_r - 4\pi\mu\Omega^2 e^{-\frac{\pi\Omega}{2\omega_0}} \sin\tau_0,$$

and after rescaling $H_r \to (H_r - \omega_0^2)/\omega_0^2$, the canonical mapping

$$\begin{aligned} w' &= w + W\sin\tau_0, \\ \tau_0' &= \tau_0 + \lambda \ln\frac{32}{|w'|} \qquad \mathrm{mod}(2\pi), \end{aligned} \tag{9.15}$$

$$W = -4\pi\mu\lambda^2 e^{-\pi\lambda/2}, \qquad \lambda = \frac{\Omega}{\omega_0} \sim \frac{1}{\sqrt{\epsilon}} \gg 1,$$

describes the motion in the vicinity of the separatrix.

The mapping (9.15) is known as the separatrix mapping or *whisker mapping* (WM) in the terminology introduced by Arnold, calling whiskers the different branches of the separatrix, and *whiskered torus* the unstable points. Actually, the unstable (stable) point may be considered as a separate orbit, since for initial conditions $p_0 = 0, \psi_0 = \pm\pi$ ($p_0 = 0, \psi_0 = 0$), the system remains there for an infinite time. In the pendulum model, we have two whiskers (technically, the stable and unstable manifolds), arriving to and departing from the whiskered torus. Therefore, the whiskers are asymptotic trajectories that, for $t \to \pm\infty$, approach towards the unstable point (the whiskered torus). However, as we have already discussed, this picture is better seen in higher dimensions.

It is easy to show that the WM is canonical since if we write it as

$$w' = w + f(\tau_0), \qquad\qquad \tau_0' = \tau_0 + h(w')$$

then the generating function of the map $(w, \tau_0) \to (w', \tau_0')$ is

$$\Phi(w', \tau_0) = w'\tau_0 + F(\tau_0) + H(w'), \qquad f = -\frac{\mathrm{d}F}{\mathrm{d}\tau_0}, \quad h = \frac{\mathrm{d}H}{\mathrm{d}w'}.$$

The WM describes the whiskers under a periodic perturbation. In absence of perturbation ($V = 0$) the first of (9.15) reduces to $w' = w$, giving a fixed value for the energy. Depending on the value of w ($|w| \ll 1$), the system will rotate, oscillate or move along the separatrix. Therefore, for $V = 0$, the stochastic layer does not exist. The energy w is an integral and the motion on phase space, (p, ψ), proceeds along a smooth curve of constant energy. But for $V \neq 0$, w changes with time. The arriving whisker ($w \neq 0, w' = 0$) and the departing whisker ($w = 0, w' \neq 0$) no longer coincide. Indeed, while for

$V = 0$ and $w' = w = 0$, the whiskers coincide, they split under a perturbation, the scale of this splitting being of the order of $|2W|$. The resulting motion in phase space becomes chaotic in a neighborhood of the separatrix, giving rise to the stochastic layer (see Fig. 9.2). The chaotic behavior of $p(t)$ is what we have called, in the previous chapters, motion across the layer. That motion, however, is better described by $w(t)$.

Fig. 9.2 displays an orbit in the WM using the variable $\hat{s} = w/W$ instead of w, for $\lambda = 8$ and $\mu \approx 3.5 \times 10^{-8}$. This figure shows that the variation of $\hat{s}$ is bounded, the width of the stochastic layer seems to be of the order of $|\hat{s}|_{\max} \sim 10$. We can also distinguish two regions; a central one, very close to the separatrix ($w = 0$), that looks like ergodic[1]: $|\hat{s}| \lesssim 2$, and a external one: $2 \lesssim |\hat{s}| \lesssim 10$, where the phase space is shared between stochastic and regular motion. It is important to remark that the stability domains are due to resonances between the resonant phase (ψ) and the phase of the perturbation (τ) in a neighborhood of the separatrix. This type of resonances are, in some sense, different from that discussed above. Indeed, the *first level resonances*, in this formulation are those of the form $\boldsymbol{m} \cdot \boldsymbol{\omega} \approx 0$ in (9.1), for some $\boldsymbol{m}$ where $\boldsymbol{m} \cdot \boldsymbol{\vartheta}$ is the resonant phase, while the *second level resonances*, are those between the oscillations of the resonant phase, $\psi = \boldsymbol{m} \cdot \boldsymbol{\vartheta}$ (in a vicinity of the separatrix in this case), with the phase $\boldsymbol{m}' \cdot \boldsymbol{\vartheta}$ of some other perturbing term $V_{\boldsymbol{m}'}$. These second level resonances can also be seen (in a different space) in, for instance Fig. 8.5.

Let us discuss in detail these second level resonances. We can either take the WM (9.15) or simply impose commensurability between the two involved frequencies, $\omega(w)$ and Ω. In the map, for some fixed value of w_r, τ_0 returns to the same point after one iteration of the map ($\tau_0' = \tau_0$). Therefore from the second equation in (9.15) we get

$$\tau_0' - \tau_0 = 2n\pi, \quad \rightarrow \quad \lambda \ln\left(\frac{32}{|w_r|}\right) = 2n\pi, \quad n \in \mathbb{Z}^+,$$

and therefore the set of resonances is

$$w_r^{(n)} = \pm 32 e^{-\frac{2n\pi}{\lambda}}, \qquad w_r^{(n)} < w_r^{(n-1)}. \tag{9.16}$$

Some of these resonances can be clearly seen in Fig. 9.2. In order to find out the location of the stable centers (if any) and the unstable points of this

[1]Roughly speaking, by ergodic we mean that the motion fills densely and uniformly some region of the phase space.

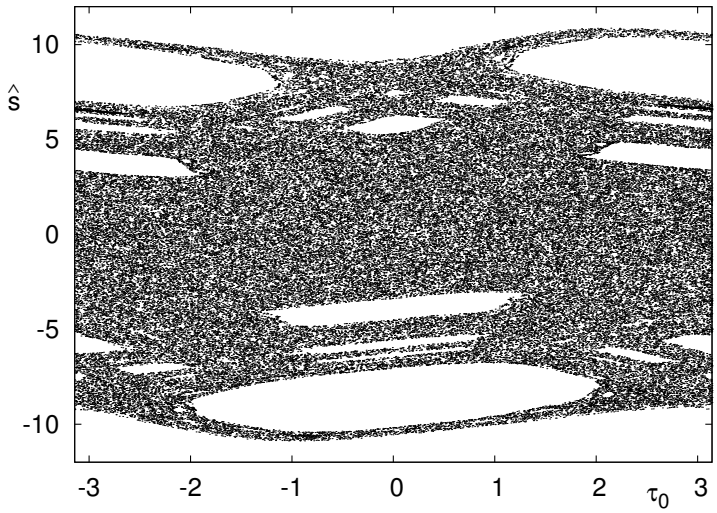

Figure 9.2: A sketch of the stochastic layer for the WM (9.15) in the $(\tau_0, \hat{s})$ plane, for $\lambda = 8$ and $\mu \approx 3.5 \times 10^{-8}$ ($W = 10^{-10}$), and $t = 4 \times 10^6$ iterations. The unperturbed separatrix in these variables lies in $\hat{s} = 0$, while $\hat{s} < 0$ and $\hat{s} > 0$ corresponds to oscillations and rotations, respectively.

resonance set, with the help of the first in (9.15) and since $w' = w = w_r^{(n)}$,

$$W \sin \tau_0 = 0, \quad \rightarrow \quad \tau_0 = k\pi, \qquad k = 0, \pm 1.$$

Thus the fixed points in the WM are $\{(w, \tau_0) : (w, \tau_0) = (w_r^{(n)}, k\pi)\}$. By inspecting Fig. 9.2 it is possible to identify the stable and unstable fixed points for those resonant values $|w_r|$ away from $w \approx 0$.

On the other hand, looking for resonances between the two involved frequencies, $\omega(w)$ and Ω, we should write

$$m\Omega - 2n\omega(w_r) = 0, \qquad m, n \in \mathbb{Z},$$

where the factor 2 appears due to the fact that, as defined, $\omega(w) = \pi/T$, with T the half-period of oscillation. Recalling the expression for $\omega(w)$ given in (9.2), the above condition implies

$$m\Omega = 2\pi n \frac{\omega_0}{\ln(32/|w_r|)}, \quad \rightarrow \quad w_r^{(m,n)} = 32 e^{-\frac{2n\pi}{m\lambda}},$$

which comparing with (9.16) $m = 1$ should be taken in $w_r^{(m,n)}$ since $w_r^{(n)}$ was derived for the situation in which τ returns to its initial value after one iteration of the map, being Ω its frequency.

These resonant values will help to understand the structure of the stochastic layer as shown in Fig. 9.2. Let us take any $w_r^{(n)}$ which for simplicity we denote it by w_r and since $|W| \ll 1$ (because $\lambda \gg 1$), we can linearize the WM in w around this resonant value w_r. Then the second term in the right-hand side in the equation for τ_0 - see (9.15), takes the form

$$\ln\left(\frac{32}{|w'|}\right) = \ln 32 - \ln|w'| \approx \ln 32 - \ln|w_r| - \frac{1}{|w_r|}(w' - w_r),$$

thus

$$\lambda \ln\left(\frac{32}{|w'|}\right) \approx \underbrace{\lambda \ln\left(\frac{32}{|w_r|}\right)}_{2n\pi} - \underbrace{\frac{\lambda}{|w_r|}(w' - w_r)}_{-\mathcal{I}'},$$

where we have used the resonance condition (9.16) and introduced a rescaled "energy" $-\mathcal{I}$. Then after this linearization the equation for τ_0 given in (9.15)

$$\tau_0' = \tau_0 + \lambda \ln\frac{32}{|w'|}$$

takes the form

$$\tau_0' = \tau_0 + \mathcal{I}',$$

after neglecting the term $2n\pi$ since τ_0 is mod 2π. The equation for w

$$w' = w + W \sin\tau_0$$

after replacing $w = w_r - \mathcal{I}|w_r|/\lambda$ reads

$$\mathcal{I}' = \mathcal{I} - \frac{\lambda W}{|w_r|}\sin\tau_0,$$

and defining

$$\kappa = -\frac{\lambda W}{|w_r|} > 0, \tag{9.17}$$

the linearized map reduces to

$$\begin{aligned} \mathcal{I}' &= \mathcal{I} + \kappa \sin \tau_0, \\ \tau_0' &= \tau_0 + \mathcal{I}' \end{aligned} \tag{9.18}$$

Therefore a SM represents the dynamics in a neighborhood of any resonance $w_r^{(n)}$, with a parameter κ that depends on the global parameters of the WM but also on the local value $w_r = w_r^{(n)}$, for $n = 0, 1, 2, \ldots$.

We know from our study of the SM that if $\kappa \gtrsim 1$ the motion is mostly chaotic or instable. Therefore if $\kappa < 1$ the the dynamics results essentially stable, several invariant tori bound stable motion and then for $|w| \approx |w_r^{(n)}| > \lambda|W|$ the map (9.18) would not lead to any strong chaotic motion; far away from the unperturbed separatrix ($w = 0$) the stochastic layer looks regular, therefore it is possible to define a rough size of the chaotic layer, w_s, by setting $\kappa \approx 1$ and then

$$|w| \lesssim w_s \approx \lambda|W|,$$

where w_s is half-width of the layer. If we compare this estimate with the numerical example given in Fig. 9.2, where instead of w it is plotted $w/|W|$ and for the given parameters of the WM, $|w|/|W| \lesssim \lambda$ is an acceptable first approximation[2].

Let us go further and consider the full resonance set (9.16). The separation between the $(n-1)$ and (n)-resonances is

$$\Delta_n = |w_r^{(n-1)} - w_r^{(n)}| = 32\left(e^{-\frac{2(n-1)\pi}{\lambda}} - e^{-\frac{2n\pi}{\lambda}}\right) = 32e^{-\frac{2n\pi}{\lambda}} \underbrace{\left(e^{\frac{2\pi}{\lambda}} - 1\right)}_{\approx 2\pi/\lambda},$$

$$\Delta_n \approx \frac{64\pi}{\lambda}\bar{w}_1^n,$$

where

$$\bar{w}_1 \equiv \frac{w_r^{(1)}}{32} = e^{-\frac{2\pi}{\lambda}} < 1.$$

Therefore since resonances accumulate towards $w = 0$ (see (9.16)), and their separation decreases as n increases, we may infer that close to the center of the layer (around the unperturbed separatrix) the motion should be extremely chaotic, like ergodic, due to the massive overlap of resonances. This

[2]This estimate could be improved leading to $\lambda \lesssim |w|/|W| \lesssim \lambda + 2$.

can also be derived from (9.17) and (9.18). Indeed, since $\kappa = \lambda|W|/|w_r| \approx w_s/|w_r|$, then if $w \approx |w_r| \ll w_s$ the parameter of the local SM (κ) that describes the WM close to $w = 0$ becomes very large, and therefore the motion in the central part of the stochastic layer should be rather chaotic.

Recall that in Fig. 8.5 only some numerical experiments regarding the SM are shown, the largest K value considered was $K = 2.5$, where the phase space looks like ergodic with the exception of few domains of stability corresponding to the centers of the integer resonances. It is not difficult to show (see next chapter) that the stable fixed point of the SM corresponding to the integer resonances becomes unstable for $K = 4$, and it is easy to check this by a simple numerical experiment. Thus for $K \geq 4$ the motion in the SM could be considered as nearly ergodic.

From the above discussion it is expected that the motion around resonances in the stochastic layer with $\kappa \gtrsim 4$ seems to be ergodic, this implies $|w_r| \lesssim w_s/4$.

All the above discussion and estimates allow us to explain the general structure of the stochastic layer, as for instance, the example given in Fig. 9.2. It is important to remark that we have derived the WM to understand the motion near the separatrix of a non-linear resonance. The latter is well represented by a SM with some "global" parameter K. And the motion close to any resonance of the WM could be modelled by a local SM defined by the parameter κ. Clearly, these two parameters are completely different, since as we discussed above, the original SM (K) describes a resonance whose resonant phase is $\psi = \boldsymbol{m} \cdot \boldsymbol{\vartheta}$, while the local SM ($\kappa$) represents a resonance with a slow phase like $(\psi - \boldsymbol{m'} \cdot \boldsymbol{\vartheta})$ near the separatrix of, for instance, an integer resonance of the SM (K). It turns out that in the SM (κ), the motion close to the separatrix of any resonance like $\mathcal{I}_r = 2n\pi$, could also be studied by means of a new WM with a parameter $\mathcal{W}$. It becomes clear that this sequence would be repeated again and again to reach any level of resonances.

To end this section let us recall that (9.14) provides the change of the unperturbed integral H_r under the effect of a small perturbation μV. The latter local integral is just the pendulum model for a given resonance of the Hamiltonian (9.1) of amplitude $\varepsilon V_{\boldsymbol{m}}$, and thus we adopted for the small oscillation frequency $\omega_0 \sim \sqrt{\varepsilon}$. If we set $\mu V \to \varepsilon V_{\boldsymbol{m'}}$, being $\boldsymbol{m'}$ another harmonic of (9.1) with a slow phase (but not resonant as the harmonic $\boldsymbol{m}$) (9.14) takes the form

$$\Delta H_r \propto \mu e^{-1/\sqrt{\varepsilon}} \sim \varepsilon e^{-1/\sqrt{\varepsilon}},$$

showing that the variation of H_r is exponentially small. This order is unreachable by means of the asymptotic series technique. Indeed, $\exp(-1/\sqrt{\varepsilon})$ is not analytic and it does not admit any power expansion in the perturbation parameter ε.

9.2 The evaluation of the Melnikov-Arnold Integral

Let us recall that the Melnikov-Arnold Integral (MAI) given by (9.9) is

$$A_m(\lambda) = \int_{-\infty}^{\infty} \cos\left(2m \arctan \mathrm{e}^t - \frac{m\pi}{2} - \lambda t\right) \mathrm{d}t \qquad m \in \mathbb{Z}^+, \tag{9.19}$$

that could be written as

$$A_m(\lambda) = \int_{-\infty}^{\infty} \exp i \left(2m \arctan \mathrm{e}^t - \frac{m\pi}{2} - \lambda t\right) \mathrm{d}t, \tag{9.20}$$

since

$$\int_{-\infty}^{\infty} \sin\left(2m \arctan \mathrm{e}^t - \frac{m\pi}{2} - \lambda t\right) \mathrm{d}t = 0,$$

due to the odd character of the argument of the sinus. Now let us express (9.20) as

$$A_m(\lambda) = \int_{-\infty}^{\infty} \exp im \left(2 \arctan \mathrm{e}^t - \frac{\pi}{2}\right) \mathrm{e}^{-i\lambda t} \mathrm{d}t, \tag{9.21}$$

and rewrite the argument of the first exponential in (9.21) in a different way. Notice that

$$\exp im \left(2 \arctan \mathrm{e}^t - \frac{\pi}{2}\right) = \left[\exp i \left(\underbrace{2 \arctan \mathrm{e}^t}_{z} - \frac{\pi}{2}\right)\right]^m,$$

so if

$$z = 2 \arctan \mathrm{e}^t,$$

it results

$$\mathrm{e}^t = \tan\frac{z}{2} = \frac{\sin z/2}{\cos z/2} = \frac{\mathrm{e}^{iz}-1}{i\,(\mathrm{e}^{iz}+1)},$$

and solving for e^{iz}

$$\mathrm{e}^{iz} = \frac{i\mathrm{e}^t+1}{1-i\mathrm{e}^t}.$$

The above expression is almost the desired argument, but we need

$$\mathrm{e}^{i(z-\pi/2)} = -i\mathrm{e}^{iz} = \frac{i\mathrm{e}^t+1}{\mathrm{e}^t+i}.$$

Finally, (9.21) takes the form

$$A_m(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t}\left(\frac{i\mathrm{e}^t+1}{\mathrm{e}^t+i}\right)^m \mathrm{d}t. \tag{9.22}$$

This integral has infinite poles:

$$\mathrm{e}^t = -i \qquad \rightarrow \qquad t_p^{(n)} = -i\frac{\pi}{2} \pm 2ni\pi, \qquad n \in \mathbb{N}. \tag{9.23}$$

The factor $\mathrm{e}^{-i\lambda t}$ in (9.22) has a major role in the "convergence" of the MAI. Let $t = x+iy$, so $-i\lambda t = -i\lambda x+\lambda y$, then if $\lambda > 0$ we should take $y < 0$ in order to $\Re(i\lambda t) < 0$. Thus to compute the integral (9.22) we consider a path over the negative imaginary part of t. Fig. 9.3 shows that path over which the MAI will be computed. The full path is defined as in Fig. 9.3, the one over AB where $t = x+iy$ ranges from $(-T, 0)$ to $(T, 0)$ and after taking the limit when $T \to \infty$, corresponds the value of the MAI (9.22) that we want to compute

$$\lim_{T\to\infty} \int_{-T}^{T} \mathrm{e}^{-i\lambda t}\left(\frac{i\mathrm{e}^t+1}{\mathrm{e}^t+i}\right)^m \mathrm{d}t.$$

Over the path CD it is $t = x - iT$ so

$$\int_{CD} = \int_T^{-T} \mathrm{d}x \underbrace{\mathrm{e}^{-i\lambda(x-iT)}}_{\mathrm{e}^{-i\lambda x}\mathrm{e}^{-\lambda T}\to 0 \text{ when } T\to\infty} \left[\frac{1+i\mathrm{e}^t}{\mathrm{e}^t+i}\right]^m \to 0$$

whenever the path does not pass through any pole. Indeed, in this case

$$\left[\frac{1+i\mathrm{e}^x\mathrm{e}^{-iT}}{\mathrm{e}^x\mathrm{e}^{-iT}+i}\right]$$

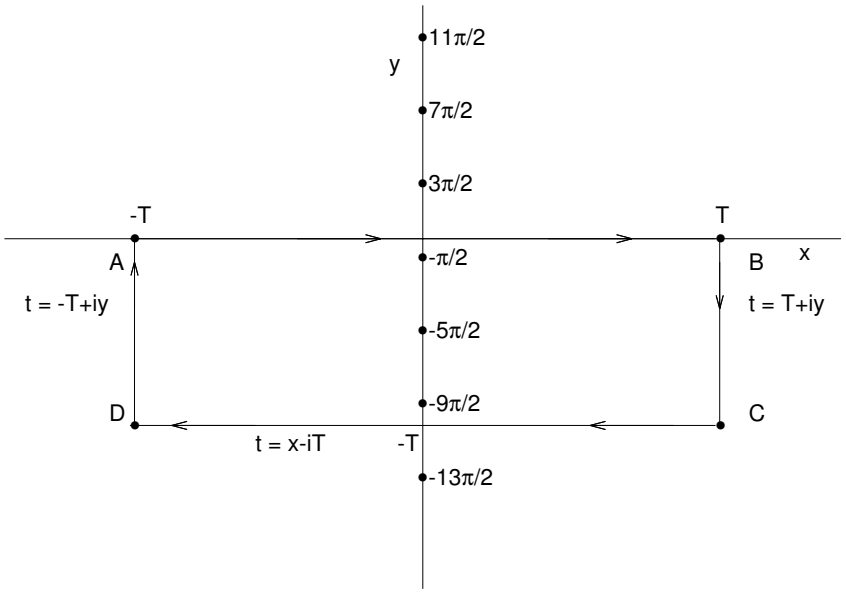

Figure 9.3: Path over the complex plane of t where the integration is done. The contour over the real axis x after $T \to \infty$ is the range of interest for the MAI. Some of the poles given by (9.25) are indicated along the y axis.

with $|\mathrm{e}^{-iT}| = 1$ is finite and then the limit when $T \to \infty$ leads to $\int_{CD} = 0$. If the path includes a pole, the limit does not exists but since the poles are isolated (the set $\{t_p^{(n)}\}$ has zero measure for in $\mathbb{R}$) it is always possible to take a path that enclose them.

Over the path BC, $t = T + iy$ and then

$$\int_{BC} = \int_0^{-T} \mathrm{d}y\, \mathrm{e}^{-i\lambda(T+iy)} \left[\frac{1 + i\mathrm{e}^{T+iy}}{\mathrm{e}^{T+iy} + i}\right]^m,$$

where

$$\lim_{T\to\infty} \left[\frac{1 + i\mathrm{e}^{T+iy}}{\mathrm{e}^{T+iy} + i}\right] = \lim_{T\to\infty} \frac{i\mathrm{e}^{T}}{\mathrm{e}^{T}} = i,$$

thus

$$\int_{BC} \approx i^m \lim_{T\to\infty} \mathrm{e}^{-i\lambda T} \int_0^{-T} \mathrm{e}^{\lambda y} \mathrm{d}y = i^m \lim_{T\to\infty} \mathrm{e}^{-i\lambda T}\, \frac{\mathrm{e}^{-\lambda T} - 1}{\lambda} \approx \frac{-1}{\lambda} \times \lim_{T\to\infty} \mathcal{O}(\lambda\mathcal{T}),$$

where $\mathcal{O}(\lambda\mathcal{T})$ are oscillatory terms of frequency λ. Therefore

$$\int_{BC}$$

does not have any limit when $T \to \infty$ since it presents oscillations with amplitudes of the order of $1/\lambda$ that are small if λ is large.

The integral over the path DA is similar to that over the BC path, therefore

$$\oint = \int_{-\infty}^{\infty} + \delta,$$

where $|\delta| \leq 1/\lambda$ is assumed to be small for large λ and thus we neglect all oscillatory terms and therefore

$$A_m(\lambda) = \oint \approx \int_{-\infty}^{\infty} \approx -2\pi i \sum_n \mathrm{Res}(t_p^{(n)}),$$

where the minus sign is introduced since the path is taken clockwise (negative direction) and $t_p^{(n)} = -i\pi/2 - 2ni\pi$, $n = 0, 1, 2, \ldots$.

Thus, we need to compute the residues on the poles. From (9.22), let

$$f(t) = \mathrm{e}^{-i\lambda t} \left(\frac{i\mathrm{e}^t + 1}{\mathrm{e}^t + i} \right)^m$$

and

$$g(t) = \left(t - t_p^{(n)}\right)^m f(t).$$

Clearly $g(t)$ is analytic and thus its Taylor expansion around any pole $t_p^{(n)}$ is

$$g(t) = g(t_p^{(n)}) + g'(t_p^{(n)})(t - t_p^{(n)}) + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}t^k}\Big|_{t=t_p^{(n)}} (t - t_p^{(n)})^k,$$

and then the Laurent series for $f(t)$ results

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}t^k}\Big|_{t=t_p^{(n)}} (t - t_p^{(n)})^{k-m}.$$

The residues are the coefficients of the Laurent series at order $(t - t_p^{(n)})^{-1}$ thus $k = m - 1$ and the residues are

$$\mathrm{Res}(t_p^{(n)}) = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}g}{\mathrm{d}t^{m-1}}\Big|_{t=t_p^{(n)}}.$$

Introducing $u = t - t_p^{(n)}$ or $t = u - i\pi/2 - 2ni\pi$, then g reads

$$g(u) = u^m \mathrm{e}^{-i\lambda u}\mathrm{e}^{-\lambda\pi/2}\mathrm{e}^{-2n\lambda\pi}\left(\frac{1+\mathrm{e}^u}{i(1-\mathrm{e}^u)}\right)^m, \tag{9.24}$$

so

$$\mathrm{Res}(t_p^{(n)}) = \frac{1}{(m-1)!}\mathrm{e}^{-\lambda\pi/2}\mathrm{e}^{-2n\lambda\pi}\frac{1}{i^m}\frac{\mathrm{d}^{m-1}}{\mathrm{d}u^{m-1}}\left[\mathrm{e}^{-i\lambda u}u^m\left(\frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right)^m\right]_{u=0}.$$

The sum over all residues is

$$\sum_{n=0}^{\infty}\mathrm{Res}(t_p^{(n)}) = \frac{1}{(m-1)!}\mathrm{e}^{-\lambda\pi/2}\frac{\mathrm{d}^{m-1}}{\mathrm{d}u^{m-1}}\left[\mathrm{e}^{-i\lambda u}u^m\left(\frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right)^m\right]_{u=0}\sum_{n=0}^{\infty}\mathrm{e}^{-2n\lambda\pi},$$

and since

$$\sum_{n=0}^{\infty}\mathrm{e}^{-2n\lambda\pi} = \sum_{n=0}^{\infty}\left(\mathrm{e}^{-2\lambda\pi}\right)^n$$

then if $|\mathrm{e}^{-2\lambda\pi}| < 1$ the above series converges to

$$\sum_{n=0}^{\infty}\mathrm{e}^{-2n\lambda\pi} = \frac{1}{1-\mathrm{e}^{-2\lambda\pi}} = \frac{\mathrm{e}^{\lambda\pi}}{2\sinh(\lambda\pi)}.$$

Therefore,

$$A_m(\lambda) \approx \frac{-2\pi i}{(m-1)!}\mathrm{e}^{-\lambda\pi/2}\frac{\mathrm{e}^{\lambda\pi}}{2i^m\sinh(\lambda\pi)}\frac{\mathrm{d}^{m-1}}{\mathrm{d}u^{m-1}}\left[\mathrm{e}^{-i\lambda u}u^m\left(\frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right)^m\right]_{u=0},$$

or

$$A_m(\lambda) \approx \frac{-\pi\mathrm{e}^{\lambda\pi/2}}{(m-1)!i^{m-1}\sinh(\lambda\pi)}\lim_{u\to 0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}u^{m-1}}\left[\mathrm{e}^{-i\lambda u}\left(u\frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right)^m\right], \quad \lambda > 0. \tag{9.25}$$

For instance

$$A_1(\lambda) = -\frac{\pi \mathrm{e}^{\lambda\pi/2}}{\sinh(\lambda\pi)} \times \lim_{u\to 0} \mathrm{e}^{-i\lambda u} u \left(\frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right),$$

and since the value of above limit is -2 we finally get

$$A_1(\lambda) = \frac{2\pi \mathrm{e}^{\lambda\pi/2}}{\sinh(\lambda\pi)}.$$

The computation of $A_2(\lambda)$ though tedious is straightforward and depends on

$$L = \lim_{u\to 0} \frac{\mathrm{d}}{\mathrm{d}} \left[\mathrm{e}^{-i\lambda u} \left(u \frac{1+\mathrm{e}^u}{1-\mathrm{e}^u}\right)^2\right],$$

so

$$A_2(\lambda) \approx \frac{-\pi}{i} \frac{\mathrm{e}^{\lambda\pi/2}}{\sinh(\lambda\pi)} \times L.$$

After performing the derivative and some algebra, L reads

$$L = \lim_{u\to 0} \mathrm{e}^{-i\lambda u}(1+\mathrm{e}^u) \left[-i\lambda \frac{u^2(1+\mathrm{e}^u)}{(1-\mathrm{e}^u)^2} + \frac{2u}{(1-\mathrm{e}^u)^3}(1-\mathrm{e}^{2u}+2u\mathrm{e}^u)\right],$$

and it is enough to take $\mathrm{e}^u = 1+u+\mathcal{O}(u^2)$ when $u \to 0$, so keeping only the linear terms in u and taking the limit it results $L = -4i\lambda$, and thus

$$A_2(\lambda) = 2\lambda A_1(\lambda).$$

There is a recurrence relation for $A_m(\lambda)$ derived by Zhirov,

$$A_{m+1}(\lambda) = \frac{2\lambda}{m} A_m(\lambda) - A_{m-1}(\lambda). \tag{9.26}$$

Now, if we consider $\lambda < 0$, the path to compute the MAI should be taken over the positive imaginary plane anti-clockwise such that $\lambda y < 0$. Let us evaluate $A_m(-\lambda)$ with $\lambda > 0$, taking now the poles on the positive imaginary axis: $t_p^{(n)} = i3\pi/2 + 2ni\pi, n = 0, 1, 2, \ldots$.

Changing $\lambda \to -\lambda > 0$ and $u \to -u$ in (9.24)

$$g = \left(\frac{-1}{i}\right)^m u^m \mathrm{e}^{-i\lambda u} \mathrm{e}^{-\lambda\pi} \mathrm{e}^{-\lambda\pi/2} \mathrm{e}^{-2n\lambda\pi} \left(\frac{1+\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\right)^m,$$

or, since

$$\left(\frac{1+\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\right)^m = (-1)^m \left(\frac{1+\mathrm{e}^{u}}{1-\mathrm{e}^{u}}\right)^m$$

it results

$$g(-u,-\lambda) = \mathrm{e}^{-\lambda\pi} g(u,\lambda).$$

Changing the sign of the derivative with respect to u in (9.25), a factor $(-1)^{m-1}$ should be included, and since the path is taken in the positive sense, an additional factor -1 in front of (9.25) is required, leading to

$$A_m(-\lambda) = (-1)^m A_m(\lambda)\mathrm{e}^{-\lambda\pi}, \qquad \lambda > 0. \tag{9.27}$$

Thus we note that $|A_m(-\lambda)| \ll |A_m(\lambda)|$ if λ is large. Moreover for $\lambda \gg 1$ $\sinh(\lambda\pi) \approx \mathrm{e}^{\lambda\pi/2}$ so, for instance,

$$A_1(\lambda) = 4\pi\mathrm{e}^{-\lambda\pi/2},$$

and in particular for $\lambda \gg m$ the recurrence relation (9.26) yields

$$A_{m+1}(\lambda) \approx \frac{2\lambda}{m} A_m(\lambda),$$

or

$$A_m(\lambda) \approx \frac{(2\lambda)^{m-1}}{(m-1)!} A_1(\lambda) \approx \frac{4\pi(2\lambda)^{m-1}}{(m-1)!}\mathrm{e}^{\lambda\pi/2}.$$

Chapter 10

Chaotic motion

We have already discussed that any perturbation to a resonance (or pendulum model) leads to chaotic motion. Depending on the strength of the perturbation the corresponding instability would be large (overlap of resonance) or small (confined to the stochastic layer of resonances). We wonder which is the signature of this type of irregular, stochastic dynamics. In order to discuss this, let us recall that the dynamics of a non-linear system is understood as resonances interaction. Thus as we have mentioned, a simplified but illustrative picture of phase space when the perturbation is not too small would be for instance, that presented in Fig. 8.5. In general, any non-linear system exhibits a divided phase space, one where the motion is stable, ordered or regular and another where the motion is unstable, chaotic. Let us then take again the SM

$$\begin{aligned} I' &= I + K \sin\vartheta, \\ \vartheta' &= \vartheta + I', \end{aligned} \tag{10.1}$$

and consider two nearby initial conditions (I_0, ϑ_0) and (I_1, ϑ_1), where $|I_1 - I_0| \ll 1, |\vartheta_1 - \vartheta_0| \ll 1$. Each set of initial conditions leads to two different orbits of the SM, say

$$\begin{aligned} &(1) \qquad I' = I + K\sin\vartheta, \qquad \vartheta' = \vartheta + I' \quad \text{for} \quad (I_0, \vartheta_0), \\ &(2) \qquad I^{*\prime} = I^* + K\sin\vartheta^*, \qquad \vartheta^{*\prime} = \vartheta^* + I^{*\prime} \quad \text{for} \quad (I_1, \vartheta_1). \end{aligned}$$

The difference of this two initially nearby orbits (2) − (1) :

$$\xi = \vartheta^* - \vartheta, \qquad \eta = I^* - I$$

evolves according to

$$\begin{aligned} \eta' &= \eta + K(\sin\vartheta^* - \sin\vartheta), \\ \xi' &= \xi + \eta'. \end{aligned}$$

Since $|I_1 - I_0| \ll 1, |\vartheta_1 - \vartheta_0| \ll 1$, we focus on the evolution of the difference orbit while $|\xi| \ll 1, |\eta| \ll 1$. Then

$$\sin\vartheta^* = \sin\vartheta + (\vartheta^* - \vartheta)\cos\vartheta + \cdots = \sin\vartheta + \xi\cos\vartheta + \mathcal{O}(\xi^2),$$

and taking only the linear part, we arrive to the so-called *tangent map*

$$\begin{aligned} \eta' &= \eta + (K\cos\vartheta)\xi, \\ \xi' &= \xi + \eta'. \end{aligned} \tag{10.2}$$

This linear map (in ξ, η) is local and describes the motion of a small section of the phase space around the SM orbit (1). Indeed, its local character is clear since to compute (10.2) it is necessary to know the solution of the SM to get $\vartheta(t)$ for the initial condition of the orbit (1) in order to evaluate the time dependent coefficient $(K\cos\vartheta(t))$ in the first of (10.2). And due to this fact it is not easy to analytically deal with both the SM and its tangent map. Thus let us consider a simpler one, that will help to understand (qualitatively) the general behavior of the SM. Consider the canonical map $(p, x) \to (p', x')$ defined through

$$\begin{aligned} p' &= p + kx, \\ x' &= x + p', \qquad x \bmod 1, \end{aligned} \tag{10.3}$$

where k is a constant parameter. This map is not linear since x being mod 1 has a discontinuity at $x = 0$. It is very simple to derive the corresponding

tangent map to (10.3), thus, taking the first variation of each equation and setting $\delta p = \eta$ and $\delta x = \xi$, then

$$\begin{aligned} \eta' &= \eta + k\xi, \\ \xi' &= \xi + \eta', \end{aligned} \tag{10.4}$$

which in fact is linear in both ξ and η and its coefficients are constant. Note that this map is global since it does not depend on any particular solution to (10.3). We rewrite (10.4) as

$$\xi' = (k+1)\xi + \eta,$$

$$\eta' = k\xi + \eta,$$

where introducing the deviation vector $\boldsymbol{l}$ and the matrix Λ as

$$\boldsymbol{l} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \Lambda = \begin{pmatrix} k+1 & 1 \\ k & 1 \end{pmatrix},$$

the map (10.4) takes the simple form

$$\boldsymbol{l}' = \Lambda \boldsymbol{l}, \tag{10.5}$$

where $\det(\Lambda) = 1$. The stability of the motion in the original map (10.3) is determined by the eigenvalues of Λ, λ, defined by

$$\det(\Lambda - \lambda I) = 0,$$

or

$$\begin{vmatrix} k+1-\lambda & 1 \\ k & 1-\lambda \end{vmatrix} = 0,$$

that leads to[1]

$$\lambda^2 - (k+2)\lambda + 1 = 0.$$

[1]The well known result $\lambda^2 - \mathrm{Tr}(\Lambda)\lambda + \det(\Lambda) = 0$, where $\mathrm{Tr}(\Lambda)$ and $\det(\Lambda)$ are invariant under a change of basis.

The two different eigenvalues are then

$$\lambda_\pm = 1 + \frac{k}{2} \pm \sqrt{k\left(1+\frac{k}{4}\right)}, \qquad \lambda_+\lambda_- = \det(\Lambda) = 1, \quad \lambda_+ + \lambda_- = \mathrm{Tr}(\Lambda). \tag{10.6}$$

The corresponding eigenvectors $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ for λ_+ and λ_- respectively, have as components in the original basis $\mathcal{B} = \{\boldsymbol{n}_1, \boldsymbol{n}_2\}$

$$e_+ = \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix}, \qquad e_- = \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}, \tag{10.7}$$

and satisfy

$$\Lambda \boldsymbol{e}_\pm = \lambda_\pm \boldsymbol{e}_\pm.$$

Then from (10.5) and the eigenvector equation we get for the components $\xi_\pm, \eta_\pm$

$$\frac{\eta_\pm}{\xi_\pm} = \frac{k}{\lambda_\pm - 1}.$$

Now if we introduce a (local) change of basis from $\mathcal{B} = \{\boldsymbol{n}_1, \boldsymbol{n}_2\} \to \bar{\mathcal{B}} = \{\boldsymbol{e}_+, \boldsymbol{e}_-\}$, the matrix Λ in $\bar{\mathcal{B}}$ takes the form

$$\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{10.8}$$

In basis $\mathcal{B}$,

$$\boldsymbol{l} = \xi \boldsymbol{n}_1 + \eta \boldsymbol{n}_2,$$

and let

$$\boldsymbol{l} = u\boldsymbol{e}_+ + v\boldsymbol{e}_-$$

in basis $\bar{\mathcal{B}}$. From (10.5), $\boldsymbol{l}' = \Lambda \boldsymbol{l}$, is independent of the election of the basis, thus in $\bar{\mathcal{B}}$ it reads

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \to \qquad \begin{matrix} u' = \lambda_+ u \\ v' = \lambda_- v. \end{matrix} \tag{10.9}$$

Taking an initial condition $\boldsymbol{l}_0 = u_0\boldsymbol{e}_+ + v_0\boldsymbol{e}_-$ after t iterations of the map (10.4) or (10.5), it is

$$u_1 = \lambda_+ u_0, \qquad u_2 = \lambda_+ u_1 = \lambda_+^2 u_0, \quad \ldots, \quad u_t = \lambda_+^t u_0,$$

and similarly for v. Thus

$$\begin{aligned} u(t) &= \lambda_+^t u_0 \\ v(t) &= \lambda_-^t v_0. \end{aligned} \tag{10.10}$$

As expected the time evolution of $\boldsymbol{l}$ strongly depends on the eigenvalues $\lambda_\pm$. From (10.6), it is simple to see that

$$\text{if} \qquad -4 < k < 0 \qquad \lambda_\pm \in \mathbb{C}, \qquad \lambda_+ = \lambda_-^*,$$

and since $\lambda_+\lambda_- = 1$, it follows that $|\lambda_\pm| = 1$. Then we have

$$\lambda_+ = e^{i\varpi}, \qquad \lambda_- = e^{-i\varpi}, \qquad \varpi \in S^1,$$

and therefore

$$\begin{aligned} u(t) &= u_0 \;\; e^{i\varpi t} \\ v(t) &= v_0\, e^{-i\varpi t}. \end{aligned} \tag{10.11}$$

From the above relations it turns clear that for $-4 < k < 0$, the motion is stable; for any t nearby orbits remain close to the initial condition in the tangent map, $u(t), v(t)$ are just a small oscillations around the origin ($u = 0, v = 0$), where u_0, v_0 are the corresponding small amplitudes. Therefore, since u, v are the components of the deviation vector $\boldsymbol{l}$, we ensure the stability of the motion in the original map (10.3), being u, v the evolution of a bundle of nearby orbits around a given orbit of the latter. Indeed, for any orbit $\gamma(t)$ in (10.3), nearby orbits to the latter evolve with time oscillating around it, since $\|\boldsymbol{l}\|$ measures the time evolution of a small neighborhood around $\gamma(t)$. Now, in (10.6),

$$\text{if} \qquad k < -4 \qquad \text{or} \qquad k > 0, \qquad \lambda_\pm \in \mathbb{R},$$

with $\lambda_+\lambda_- = 1$. Assume that $|\lambda_-| < 1 < |\lambda_+|$, so defining

$$\sigma_+ \equiv \sigma = \ln|\lambda_+| > 0, \qquad \sigma_- = \ln|\lambda_-| = \ln 1/|\lambda_+| = -\sigma < 0,$$

we write

$$|\lambda_+| = e^{\sigma}, \qquad |\lambda_-| = e^{-\sigma},$$

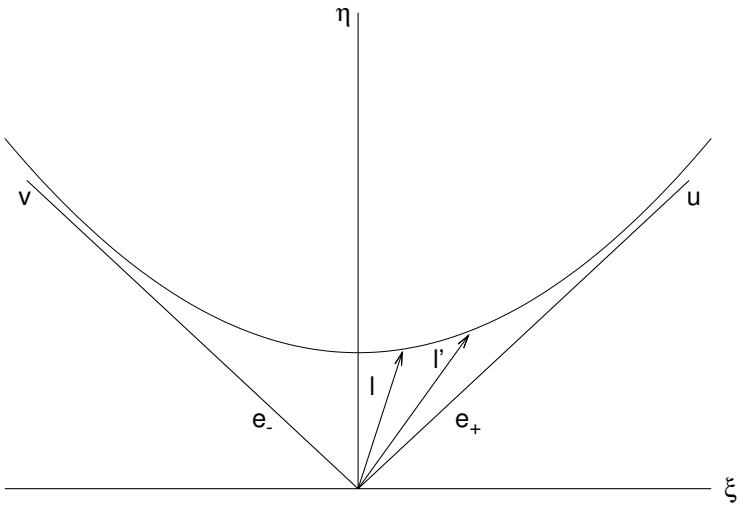

Figure 10.1: A sketch of the locally unstable motion. Nearby orbits diverge exponentially. The deviation vector $\boldsymbol{l}$ tends to the $\boldsymbol{e}_+$ direction when $t \to \infty$. The solid curve corresponds to the solution of the tangent map.

and therefore

$$\begin{aligned} u(t) &= u_0 e^{\sigma t} \\ v(t) &= v_0 e^{-\sigma t}. \end{aligned} \tag{10.12}$$

The solution (10.12) shows that the motion is doubly asymptotic, that means

$$|u(t)| \to \infty \quad \text{when} \quad t \to \infty \qquad \text{and} \qquad |u(t)| \to 0 \quad \text{when} \quad t \to -\infty,$$

$$|v(t)| \to 0 \quad \text{when} \quad t \to \infty \qquad \text{and} \qquad |v(t)| \to \infty \quad \text{when} \quad t \to -\infty.$$

Thus the motion of the original map (10.3) is unstable and nearby orbits in it diverge exponentially, as Fig. 10.1 shows. The rate at which they diverge is $\sigma = \ln|\lambda_+|$. Indeed, for t large enough, the deviation vector $\boldsymbol{l}$ becomes almost parallel to $\boldsymbol{e}_+$ and thus $\boldsymbol{l} \approx u(t)\boldsymbol{e}_+$ and then

$$\|\boldsymbol{l}\| = l_0 e^{\sigma t}, \qquad t \to \infty, \tag{10.13}$$

where $l_0 \approx u_0$. This exponential rate $\sigma > 0$ at which nearby orbits diverge is usually known as *Lyapunov exponent*, that in this particular case coincides with the so-called *Krylov-Kolmogorov-Sinai Entropy*. The motion in the original map is said hyperbolic when this exponential divergence occurs. It is easy to check from (10.12) that $uv = u_0 v_0$, which is the equation of an hyperbola in the $u - v$ or $\xi - \eta$ plane. Clearly σ has dimension of time, thus its inverse

$$T_L = \frac{1}{\sigma} \tag{10.14}$$

is called *Lyapunov time* and provides a time-scale for the manifestation of the exponential instability.

The very signature of chaotic motion is the local exponential divergence of nearby orbits. Though in the example here discussed, the map (10.3), this behavior is global (provided that k lies outside the stability domain $-4 < k < 0$), since its tangent map (10.5) does not depend on any particular solution of (10.3). However, if we go back to the SM (10.1) and its tangent map (10.2), the latter not only depends on the parameter K but also on the initial conditions in the SM that lead to the orbit $I(t), \vartheta(t)$. This is the general situation in most non-linear systems and thus we speak about the local instability of the motion around a given orbit of the system. On the other hand, if for a given initial condition the motion proceeds on a torus, we speak about the local stability of the motion and no exponential divergence takes place, as we see from (10.11). In this direction we say that unstable chaotic motion is *sensitive to initial conditions* while stable regular motion has a rather weak dependence with them.

To illustrate this local character of the instability (or stability), let us consider some fixed points of the SM (10.1). We have already seen in Chapter 7 that for an integer resonance, $J_r = n$, the fixed points appear at $\vartheta_0 = 0, \pi$. To investigate the stability of these fixed points, let us consider the tangent map (10.4) or (10.5) with $k = K \cos \vartheta_0$, where we assume $K > 0$. This map (10.4) with such parameter value is identical to the tangent map (10.2) of the SM for two different orbits (fixed points in this case). For $\vartheta_0 = 0$ the parameter $k = K > 0$ and thus the fixed point $J_r = n, \vartheta_0 = 0$ is alway unstable for positive K (it lies on the separatrix of the integer resonance). On the other hand, for $\vartheta_0 = \pi$, it is $k = -K < 0$ and the point $J_r = n, \vartheta_0 = \pi$ is stable when $0 < K < 4$ but it becomes unstable for $K > 4$[2].

[2]This result justifies the assertion that the centers of integer resonances of the SM become unstable for $K > 4$.

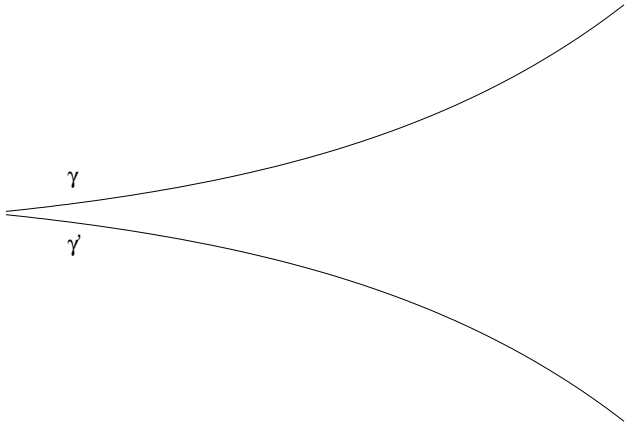

Figure 10.2: Illustration of the exponential divergence of two nearby orbits, γ and γ'.

The exponential divergence of nearby orbits given by (10.13) is schematically represented in Fig. 10.2, where initially $\gamma(t)$ and $\gamma'(t)$ are very close to each other, $\|\gamma'(t_0) - \gamma(t_0)\| = \delta_0 \ll 1$. After a motion time $t \gg t_0$

$$\delta(t) \equiv \|\gamma'(t) - \gamma(t)\| \approx \delta_0 e^{\sigma_\gamma t}, \tag{10.15}$$

where σ_γ is the Lyapunov exponent for γ. Why does this smooth, regular exponential rate of separation, lead to unstable, chaotic motion? The key point is that the motion in phase space is always bounded, thus two nearby chaotic orbits could diverge at most up to the size of the allowed region. For instance, as we have already discussed, the chaotic layer has a finite width because invariant curves of regular motion confine the chaotic motion to a narrow region around the separatrix. Thus the motion in the chaotic layer is bounded and therefore two nearby orbits could diverge at most up to its width w_s. The same occurs to the SM. The phase space is bounded since ϑ is restricted to a bounded interval, $(0, 2\pi)$. Thus the chaotic motion is always "oscillatory" in a broad sense. Therefore two nearby orbits would diverge up to the order of the size of the bounded region of chaotic motion. Then, since $\delta(t)$ could not increase any longer, initially nearby orbits (or fully correlated initial orbits) start to mix, they lose their correlations, and for t large enough, the motion looses forever its memory, that is, forgets the initial conditions. When the exponential divergence sets up, the motion seems to be completely

uncorrelated, "random", the time-scale for mixing being the Lyapunov time. This is the very physical meaning of the Lyapunov exponent or its inverse T_L.

Also stable orbits do diverge, but in a linear way. This fact is due to the frequency difference of nearby tori. To illustrate this let us consider an orbit on a given torus defined by the action $\boldsymbol{I}$, where the frequency is $\boldsymbol{\omega}(\boldsymbol{I})$. Taking on this torus an initial condition $\boldsymbol{\vartheta}_0$, then the orbit is

$$\boldsymbol{\vartheta}(t) = \boldsymbol{\omega}(\boldsymbol{I})t + \boldsymbol{\vartheta}_0. \tag{10.16}$$

Consider a nearby tori defined by $\boldsymbol{I}'$ such that $\|\boldsymbol{I}' - \boldsymbol{I}\| \ll 1$. The motion proceeds in this torus with frequency $\boldsymbol{\omega}(\boldsymbol{I}')$, and assuming initial condition $\boldsymbol{\vartheta}'_0$, the motion in this nearby tori is

$$\boldsymbol{\vartheta}'(t) = \boldsymbol{\omega}(\boldsymbol{I}')t + \boldsymbol{\vartheta}'_0, \tag{10.17}$$

setting

$$\boldsymbol{\delta I} = \boldsymbol{I}' - \boldsymbol{I}, \qquad \|\boldsymbol{\delta I}\| \ll 1,$$

the change in the frequency is therefore

$$\boldsymbol{\omega}(\boldsymbol{I}') = \boldsymbol{\omega}(\boldsymbol{I} + \boldsymbol{\delta I}) = \boldsymbol{\omega}(\boldsymbol{I}) + \boldsymbol{\delta\omega}, \qquad \delta\omega_i = \frac{\partial \omega_i}{\partial I_j}\delta I_j,$$

where in the second equality the sum over j is understood, then (10.17) can be recast as

$$\boldsymbol{\vartheta}'(t) = (\boldsymbol{\omega}(\boldsymbol{I}) + \boldsymbol{\delta\omega})t + \boldsymbol{\vartheta}'_0 = \boldsymbol{\vartheta}(t) + \boldsymbol{\delta\omega}t + \boldsymbol{\vartheta}'_0 - \boldsymbol{\vartheta}_0,$$

and defining $\boldsymbol{\delta\vartheta} = \boldsymbol{\vartheta}' - \boldsymbol{\vartheta}$ we write

$$\boldsymbol{\delta\vartheta} = \boldsymbol{\delta\omega}t + \boldsymbol{\delta\vartheta}_0. \tag{10.18}$$

This result is usually called phase mixing[3], since the phase difference of orbits laying in nearby tori grows linearly with time. For instance, assume that $\boldsymbol{\vartheta}'_0 = \boldsymbol{\vartheta}_0$, both orbits start at the very same initial phase value but their difference grows with time and when $t \to \infty$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}'$ become completely

[3]It should not be confused with the mixing process discussed above, that take place over some region of the phase space.

uncorrelated. This is a typical situation in a non-linear system. Note that in a linear one $\boldsymbol{\delta\omega} = 0$ and therefore the phases evolve in a synchronous way.

Let us consider now a Hamiltonian of the form

$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2} + V(\boldsymbol{q}), \qquad \boldsymbol{q},\boldsymbol{p} \in G \subset \mathbb{R}^N, \tag{10.19}$$

and assume that in some open domain $D \subset G$, a canonical transformation $(\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{I},\boldsymbol{\vartheta})$ exists, such that $H(\boldsymbol{q},\boldsymbol{p}) \to \mathcal{H}(\boldsymbol{I})$. In other words, within D, the motion lies on invariant tori, and in any torus an orbit is given by (10.16). For an initial condition $(\boldsymbol{q}_0,\boldsymbol{p}_0)$, the Hamiltonian flow induced by (10.19) leads to an orbit $\gamma(t) = \{\boldsymbol{q}(t),\boldsymbol{p}(t) : \boldsymbol{q}(0) = \boldsymbol{q}_0, \boldsymbol{p}(0) = \boldsymbol{p}_0\}$, over the energy surface. Any coordinate $q_i \equiv q$ of γ admits a Fourier expansion that we assume to depend only on the frequency $\omega_i(\boldsymbol{I}) \equiv \omega(\boldsymbol{I})$,

$$q(t) = \sum_n q_n(\boldsymbol{I}) \cos(n\omega(\boldsymbol{I})t),$$

and since $p_i(t) \equiv p(t) = \dot{q}(t)$

$$p(t) = -\omega \sum_n n q_n(\boldsymbol{I}) \sin(n\omega(\boldsymbol{I})t).$$

Let us take a slightly different initial condition on a nearby torus defined by $\boldsymbol{I}' = \boldsymbol{I} + \boldsymbol{\delta I}$, such that $\|\boldsymbol{\delta I}\| \ll 1$, then the coordinate $q_i' \equiv q'$ for the nearby orbit $\gamma'(t)$ has the Fourier expansion

$$q'(t) = \sum_n q_n(\boldsymbol{I}') \cos(n\omega(\boldsymbol{I}')t) = \sum_n q_n(\boldsymbol{I} + \boldsymbol{\delta I}) \cos(n\omega(\boldsymbol{I} + \boldsymbol{\delta I})t).$$

Now, at first order in $\|\boldsymbol{\delta I}\|$

$$\omega(\boldsymbol{I} + \boldsymbol{\delta I}) = \omega(\boldsymbol{I}) + \delta\omega(\boldsymbol{I}) = \omega(\boldsymbol{I}) + \frac{\partial\omega}{\partial I_j}\delta I_j,$$

$$q_n(\boldsymbol{I} + \boldsymbol{\delta I}) = q_n(\boldsymbol{I}) + \frac{\partial q_n}{\partial I_j}\delta I_j,$$

where the sum over j is understood. Assuming $\|\boldsymbol{\delta I}\|t \ll 1$

$$\cos\left(n\omega(\boldsymbol{I})t + n\frac{\partial\omega}{\partial I_j}\delta I_j t\right) = \cos(n\omega(\boldsymbol{I})t) - \sin(n\omega(\boldsymbol{I})t) n\frac{\partial\omega}{\partial I_j}\delta I_j t,$$

then at first order for q'

$$\begin{aligned} q'(t) &= q(t) + \delta I_j \sum_n \frac{\partial q_n}{\partial I_j} \cos(n\omega(\boldsymbol{I})t) - t\frac{\partial \omega}{\partial I_j}\delta I_j \underbrace{\sum_n n q_n(\boldsymbol{I}) \sin(n\omega(\boldsymbol{I})t)}_{-p(t)/\omega} \\ &= q(t) + \delta I_j \left\{ \frac{p(t)}{\omega}\frac{\partial \omega}{\partial I_j} t + \sum_n \frac{\partial q_n}{\partial I_j} \cos(n\omega(\boldsymbol{I})t) \right\}. \end{aligned}$$

Evaluating at $t = 0$

$$q'(0) - q(0) = \sum_n \frac{\partial q_n}{\partial I_j}\delta I_j = \delta q(0),$$

where $\delta q(0)$ is the slight difference in the initial condition for the coordinate q. Then the sum

$$\delta I_j \sum_n \frac{\partial q_n}{\partial I_j} \cos(n\omega(\boldsymbol{I})t) = \delta q(0) + \delta I_j \sum_{n\neq 0} \frac{\partial q_n}{\partial I_j} \cos(n\omega(\boldsymbol{I})t) = \delta q(0) + \mathrm{O}(\omega t),$$

where $\mathrm{O}(\omega t)$ is an oscillating term of small amplitude (of order $\delta I_j \ll 1$) and zero average. Neglecting then the small oscillations we get

$$q'(t) - q(t) \approx \delta q(0) \left(1 + t\frac{\delta\omega}{\omega \delta q(0)} p(t)\right),$$

introducing the linear rate of divergence

$$\lambda = \left|\frac{\delta\omega}{\omega\delta q(0)}\right| p_m > 0,$$

where p_m is the amplitude of oscillation of $p(t)$, and denoting by $\delta q(t) = |q'(t) - q(t)|$ and $\delta q_0 = |\delta q(0)|$, we finally get

$$\delta q(t) \approx \delta q_0 (1 + \lambda t). \tag{10.20}$$

Clearly the very same linear divergence would show up for any coordinate or momentum, and thus (10.20) applies for the norm of the full phase space vector, with a linear rate that could be taken as the maximum among all phase coordinates.

This different rate of divergence between stable, regular orbits and unstable chaotic ones turns out to be an efficient way to detect chaos. Indeed, let us consider again the Hamiltonian (10.19), in which for simplicity we take $N = 2$. The equations of motion are then

$$\begin{aligned} \dot{q}_1 &= p_1, \qquad \dot{p_1} = -\tfrac{\partial V}{\partial q_1} \equiv V_1, \\ \dot{q}_2 &= p_2, \qquad \dot{p_2} = -\tfrac{\partial V}{\partial q_1} \equiv V_2, \end{aligned} \tag{10.21}$$

and the first variational equations (equivalent to the tangent map) are the linear set

$$\begin{aligned} \dot{\delta q_1} &= \delta p_1, \qquad \dot{\delta p_1} = -V_{11}\delta q_1 - V_{12}\delta q_2, \\ \dot{\delta q_2} &= \delta p_2, \qquad \dot{\delta p_2} = -V_{21}\delta q_1 - V_{22}\delta q_2, \end{aligned} \tag{10.22}$$

where the derivatives V_{ij} depend on $(q_1(t), q_2(t))$ and $V_{ij} = V_{ji}$. It is clear then that we need to solve simultaneously both set of equations, (10.21) and (10.22), since the variational ones include coefficients that depend on time through the coordinates. Take an initial condition $(\boldsymbol{p}_0, \boldsymbol{q}_0)$ for (10.21), and $(\boldsymbol{\delta p}_0, \boldsymbol{\delta q}_0)$ for (10.22). Let $\gamma(t)$ be the orbit (solution of the equations of motion) for the given initial condition. The integration of the variational equations for the selected initial condition and where the coefficients V_{ij} should be evaluated at $\gamma(t)$, leads to the deviation vector

$$\boldsymbol{\delta}(t) = (\boldsymbol{\delta q}(t), \boldsymbol{\delta p}(t)),$$

which characterizes the dynamics around $\gamma(t)$. Now let us compute the following magnitude

$$\sigma^*(t) = \frac{1}{t} \ln \frac{\|\boldsymbol{\delta}(t)\|}{\|\boldsymbol{\delta}(0)\|}. \tag{10.23}$$

If $\gamma(t)$ is a regular, stable orbit, $\|\boldsymbol{\delta}(t)\|$ grows linearly with t as (10.20) and

$$\sigma^*(t) \approx \frac{\ln t}{t} \to 0, \qquad t \to \infty,$$

while if $\gamma(t)$ is chaotic, $\|\boldsymbol{\delta}(t)\|$ increases exponentially with time as (10.15), and then

$$\sigma^*(t) \to \sigma, \qquad t \to \infty$$

.

Therefore, for a regular orbit, the (maximum) Lyapunov exponent is zero while for a chaotic one is positive. The magnitude σ^* is usually called the finite time (maximum) Lyapunov exponent, and its values for different orbits allow to discriminate between stable and unstable motion.

Chapter 11

Applications

11.1 Orbits in non–axisymmetric 2D potentials

In any 2D non–axisymmetric potential the main families of orbits are the so–called loop and box. Which family dominates the orbital structure depends mainly on the relative value of the rotational kinetic energy with respect to the degree of flatness of the potential (see below). To describe the problem in a more general context, let us consider a spatial axisymmetric galaxy, where we assume that the potential depends on the position through $m_q(R,z) = R^2 + z^2/q^2$, (R,φ,z) being cylindrical coordinates and $q < 1$ the semiaxis ratio of isopotential curves in the Rz–plane. The potential is then $\phi(\boldsymbol{r}) = \Phi(m_q(R,z))$ where Φ is a smooth function of its argument.

In any case, 3D motion reduces to 2D motion in Cartesian coordinates if we introduce the effective potential $\Phi(m_q(R,z))+p_\varphi^2/2R^2$, where p_φ is a global integral. As the second term is the same for any Φ, we focus the attention on the motion of a star in the 2D potential $\phi(x,y) = \Phi(m_q(x,y))$ where x, y are coordinates in some meridian plane by setting $p_\varphi = 0$ [1]. Alternatively, $\phi(x,y)$ could represent the motion in the equatorial plane ($z = 0$) of a barlike galaxy, being then x, y coordinates in the latter plane.

[1] A value $p_\varphi \neq 0$ could change the topology of the phase space. The discussion given here could be slightly different, but the tools to study it are the same presented here. The main difference, if $p_\varphi \neq 0$, is that $R = 0$ is not admissible and box orbits do not exist.

The equations of motion, in these variables, are

$$\dot{p}_x = -2\Phi' x, \qquad \dot{p}_y = -2\Phi' y/q^2,$$

where $p_x = \dot{x}$, $p_y = \dot{y}$ and $\Phi' \equiv \mathrm{d}\Phi/\mathrm{d}m_q$ is assumed analytic everywhere. To be $-\boldsymbol{\nabla}\phi$ a well defined gravitational field it is necessary to impose the conditions $\Phi' > 0$ and $\Phi'' < 0$. To understand the differences between both families of orbits one can follow different approaches. The "rigorous" one as follows. For box orbits, one should restrict the flow to the invariant planes $p_x = x = 0$ or $p_y = y = 0$ and to investigate the 1D Hamiltonians H_y and H_x, respectively. Take one of them and consider the other as a small perturbation. The next step is to analyze the stability of the periodic orbits in the unperturbed 1D Hamiltonians, at a given energy level, by a linearization of the equations of motion of the full Hamiltonian around these orbits. Similar considerations apply to loop orbits: just take values of q very close to 1 so that the field is nearly spherical and can be written as a near integrable one. The stability of the 1:1 (circular) periodic orbit is then analyzed. This procedure is followed, for example, for the 2D logarithmic potential to conclude that, for the energies and values of q studied, the short-axis periodic orbit (y-axis) is, in general, unstable while the long-axis orbit (x-axis) is, in general, stable for low-to-moderate energies. The 1:1 periodic orbit (that bifurcates from the y–axis orbit) turns out to be always stable for any physical value of q. Therefore, box orbits can be thought as perturbations to the x-axis periodic orbits while loop orbits arise from perturbations to the 1:1 (circular) periodic orbit in the spherical system.

A physical interpretation is the following. The angular momentum (or the rotational kinetic energy) plays a crucial role in the existence of both families of orbits. Indeed, take polar coordinates in the xy-plane: $x = r\cos\theta$, $y = r\sin\theta$, so $m_q \to m_\alpha = r^2(1 + \alpha\sin^2\theta)$, where $\alpha = (1 - q^2)/q^2$. Due to the lack of central symmetry, a test star will be acted by a torque $N = -\partial\phi/\partial\theta = -\Phi'\partial m_\alpha/\partial\theta = -\alpha r^2\Phi'\sin 2\theta$. If $\Phi' \neq 0$ then for any $r > 0$ the torque is null at $\theta = 0, \pi/2$ (and $\pi, -\pi/2$), that is, on the x and y axis. Recalling that $\dot{p}_\theta = N$, where p_θ is the angular momentum of the star, we conclude that an orbit with $p_\theta = 0$ will follow a rectilinear orbit along the x or y axis. A simple inspection of the expression for N shows that the torque is negative in the first and third quadrant, being positive in the others. Therefore, we see why the x-axis periodic orbit is stable while the y-axis one is unstable. The torque confines near the x-axis and pulls away near the y–axis. On the other

hand, a simple epicycle approximation shows that the 1:1 (circular) periodic orbit is naturally stable for α not too large.

Let us recall that this description is true provided that Φ is a smooth function of m_α. If the potential has a singularity or a cusp at the origin, then the analysis may be different. Therefore the discussion given above is suitable for potentials that are not "hard" at the origin, that is, those for which the deflection angle $\Delta\theta$ is close to π when $p_\theta \to 0$.

Let us write the full Hamiltonian in polar coordinates

$$H(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \Phi\left(m_\alpha(r,\theta)\right),$$

where $p_r = \dot{r}$, $p_\theta = r^2\dot{\theta}$. Assume that α is small, that is, $1/\sqrt{2} < q \lesssim 1$, so we can expand $\Phi\left(m_\alpha(r,\theta)\right)$ in powers of α and we can separate the part independent of θ

$$\begin{aligned} H(p_r, p_\theta, r, \theta) \;&= p_r^2/2 + p_\theta^2/2r^2 + \phi_\alpha(r) - \tfrac{\alpha}{2} f_1(r)\cos 2\theta \\ &\quad - \tfrac{\alpha^2}{4} f_2(r)\left(\cos 2\theta - \tfrac{1}{4}\cos 4\theta\right) + \dots, \end{aligned} \tag{11.1}$$

where

$$\begin{aligned} &\phi_\alpha(r) \;=\; \Phi(r^2) \;+\; \tfrac{\alpha}{2}\, f_1(r) \;+\; \tfrac{3\alpha^2}{16}\, f_2(r) \;+\; \cdots, \\ &f_1(r) = \Phi'(r^2) r^2 \geq 0, \qquad f_2(r) = \Phi''(r^2) r^4 \leq 0. \end{aligned} \tag{11.2}$$

From (11.1) and (11.2) the Hamiltonian can be written as

$$\begin{aligned} &H(p_r, p_\theta, r, \theta) = H_0(p_r, p_\theta, r) + \alpha V_1 + \alpha^2 V_2 + \cdots, \\ &H_0(p_r, p_\theta, r) = \tfrac{p_r^2}{2} + \tfrac{p_\theta^2}{2r^2} + \phi_\alpha(r), \\ &V_n(r,\theta) = f_n(r) \textstyle\sum_{m=1}^{n} a_m \cos 2m\theta, \quad f_n(r) = \Phi^{(n)}(r^2) r^{2n}. \end{aligned} \tag{11.3}$$

H_0 is an integrable Hamiltonian being $H_0 = h_0$ itself and $p_\theta = p_\theta^{\mathrm{o}}$ the unperturbed integrals and $\alpha^n V_n$ are small perturbations (see the remark at the end of this subsection). So, from now on, when we refer to unperturbed motion, we mean orbits in H_0 even though it depends on α.

The unperturbed system is just a central field. So r oscillates between two boundaries, $r_m(h_0, p_\theta^{\mathrm{o}}) \leq r^{\mathrm{o}}(t) \leq r_M(h_0, p_\theta^{\mathrm{o}})$, with frequency ω_r, while θ^{o} varies on the circle S^1. The frequency in the tangential direction is $\omega_\theta = \kappa\,\omega_r$

where $\kappa = \Delta\theta/2\pi < 1$ is, in general, irrational. The time evolution of θ can be written as $\theta^{\rm o}(t) = \omega_\theta t + \Theta(t)$ where Θ is a $2\pi/\omega_r$-periodic function of time.

Let us focus the attention, in the perturbed system, on the dynamics in the tangential direction. Keeping terms up to first order in α in (11.3) we get

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \approx -\alpha \frac{\partial V_1}{\partial \theta} = -\alpha f_1(r) \sin 2\theta. \tag{11.4}$$

From (11.4) a simple manipulation shows that the latter can be written as

$$\frac{{\rm d}\mathcal{K}}{{\rm d}t} + \alpha g_1(r(t)) \sin 2\theta(t) \approx 0, \qquad \mathcal{K} \equiv \frac{p_\theta^2}{2} - \frac{\alpha}{2} g \cos 2\theta, \tag{11.5}$$

where g and $g_1(r(t))$ are the average and oscillating parts of $f_1(r(t))r^2(t)$ respectively:

$$\begin{aligned} g &= \langle\, f_1(r) r^2 \,\rangle = \langle\, \Phi'(r^2) r^4 \,\rangle > 0, \\ g_1(r(t)) &= f_1\left(r(t)\right) r^2(t) - g. \end{aligned} \tag{11.6}$$

To keep order α in the perturbation, in the second term in the first of (11.5) we can replace the actual values of r, θ by their unperturbed values $r^{\rm o}(t)$, $\theta^{\rm o}(t)$. Since the unperturbed motion is completely regular, we can expand $g_1(r^{\rm o}(t)) \sin 2\theta^{\rm o}(t)$ in Fourier series, with basic frequencies ω_r and ω_θ

$$g_1\left(r^{\rm o}(t)\right) \sin 2\theta^{\rm o}(t) = \Re \left\{ \sum_{k=-\infty}^{\infty} \tilde{g}_k {\rm e}^{i(k\omega_r + 2\omega_\theta)t} \right\}, \tag{11.7}$$

where $\tilde{g}_k$ are certain complex coefficients. Assuming quasiperiodicity (which is the more abundant behavior if α is small), i.e. κ irrational, we easily see from (11.7) that $\langle\, g_1 \sin 2\theta^{\rm o} \,\rangle \approx 0$. Hence if we average the first in (11.5) over several radial periods we see that (see below)

$$\mathcal{K} = \frac{p_\theta^2}{2} - \Omega^2 \cos 2\theta, \qquad \Omega^2 = \frac{\alpha}{2} g > 0, \tag{11.8}$$

is an approximate invariant. $\mathcal{K}$ plays the role of the total energy in a simple pendulum model where Ω is the small oscillation frequency. Therefore two critical values of $\mathcal{K}$ exist: $-\Omega^2$ and Ω^2. For $\mathcal{K} = -\Omega^2$, $(\theta, p_\theta) = (0, 0)$ is a stable equilibrium point: the motion is stable along the x axis. On the other

hand, for $\mathcal{K} = \Omega^2$ we have the separatrix and the unstable equilibrium points are $(\theta, p_\theta) = (\pm\pi/2, 0)$: the motion along the y axis is unstable. The domain of box orbits, that oscillate about the long-axis, corresponds to $|\mathcal{K}| < \Omega^2$ and the domain of loop orbits, that rotate about the origin, to $\mathcal{K} > \Omega^2$. The separatrix, $p_\theta^s = \pm 2\Omega \cos\theta^s$, separates then different kinds of motion: oscillations and rotations; i.e. box and loop orbits. For $\mathcal{K} \gg \Omega^2$, $\mathcal{K} \approx p_\theta^2/2$: the kinetic energy in the tangential direction. The largest value of $\mathcal{K}$ corresponds to the largest p_θ, which appears for the 1:1 periodic orbit. For $V_1 \neq 0$ this periodic orbit should not be circular but elliptic with small eccentricity (see below).

Since Ω is a measure of the amplitude of the torque, we conclude that the relevant parameter that defines the orbital family is the relative value of the rotational energy with respect to the strength of the torque, which in turn depends on the degree of flatness of the potential.

From the above discussion it turns out that a limit angle, θ_l, could exist

$$\cos 2\theta_l \approx -\frac{\mathcal{K}}{\Omega^2},$$

which is another way to conclude that $|\mathcal{K}| < \Omega^2$ for boxes. However it is important to remark that this bound for θ appears for r bounded away from 0. When p_θ is small, which is the case for boxes, the analysis of the motion in a neighborhood $\mathcal{R}$ of $r = 0$ should be done in a different way since the origin is a singular point in this description. As we assume that the potential is regular at $r = 0$, we can approximate $\phi(x, y)$ by a harmonic oscillator in $\mathcal{R}$. The approximate invariants of motion are then the energy in each degree of freedom h_x, h_y. But the Lissajous-like orbits in an harmonic oscillator with incommensurable frequencies are dense in $\mathcal{R}$ whenever h_x, $h_y \neq 0$, so no bound for θ exists while the star is in $\mathcal{R}$.

The frequency Ω, depends on the average $\langle f_1(r) r^2 \rangle$ over the unperturbed motion (Eqs. (11.6) and (11.8)). As defined in (11.2), $f_1(r)$ can be put in terms of the circular speed, v_c: $f_1(r) = v_c^2(r)/2$. So from (11.8) follows that

$$\Omega^2 = \frac{\alpha}{4} \langle\, v_c^2(r) r^2 \,\rangle.$$

For the realistic case of flat rotation curves at large radii we get

$$\Omega^2 \approx \frac{\alpha}{4} v_c^2 \langle\, r^2 \,\rangle \sim \frac{\alpha}{12} v_c^2 r_M^2 (1 + \beta + \beta^2), \tag{11.9}$$

where $0 \leq \beta = r_m/r_M \leq 1$, and $r_M > r_m$ are the two roots of the equation (see Eq. (11.3))

$$(p_\theta^o)^2 - 2r^2\left(h_0 - \Phi(r^2)\right) + \frac{1}{2}\alpha\Phi'(r^2)r^4 = 0,$$

which, for r_M, can be approximated by

$$(p_\theta^o)^2 - 2r_M^2\left(h_0 - \Phi(r_M^2)\right) + \frac{1}{2}\alpha v_c^2 r_M^2 \approx 0. \tag{11.10}$$

For the estimate in (11.9), where a factor 2 should be added if $\beta = 0$, we approximate the time-average of r^2 by the r–average over the allowed interval. This is not true in general but it provides a rough estimate of the average that will help us later.

The invariant $\mathcal{K}$ is in some sense local, since unperturbed orbits with different angular momentum will have different values of the frequency: $\Omega = \Omega(h_0, p_\theta^o)$. From (11.9) and (11.10) it is not difficult to conclude that the largest Ω is expected for minimum $|p_\theta^o|$; $p_\theta^o = 0$, i.e., for radial orbits, while the smallest one for maximum $|p_\theta^o|$, i.e., for circular orbits.

For the case of the 1:1 periodic orbit we can write,

$$\Omega_{1:1}^2 \approx \frac{\alpha}{4} v_c^2 a^2, \tag{11.11}$$

where a is the circular radius defined by

$$\Phi(a^2) \approx h_0 - \left(1 + \frac{\alpha}{2}\right)\frac{v_c^2}{2}. \tag{11.12}$$

Then, the maximum value of $\mathcal{K}$ lies somewhere between

$$\begin{aligned} &\frac{v_c^2 a^2}{2}\,(1 - \alpha/2) \lesssim \mathcal{K}_M \lesssim \frac{v_c^2 a^2}{2}\,(1 + \alpha/2) \\ &(2/\alpha - 1)\,\Omega_{1:1}^2 \lesssim \mathcal{K}_M \lesssim (2/\alpha + 1)\,\Omega_{1:1}^2. \end{aligned} \tag{11.13}$$

The whole picture given above is true for small α (q close to 1). Indeed, this approach makes sense when the x-axis periodic orbit is stable. It is well known, that for large values of α ($\alpha \sim 1$, $q \sim 0.7$) the latter orbit could become unstable bifurcating to another periodic orbit. A sub-family associated to this new orbit appears. It is expected also that the x-axis periodic orbit lies now in a narrow stochastic layer around the separatrix of

the resonance (see below). Other high-order resonances would occupy some region of the phase space and many zones of stochastic motion would also appear. So it is hard to speak then only about box or loop orbits when the perturbation is large (in fact, the term boxlets is often used in this case). A global study of the logarithmic potential reveals that this "very chaotic" panorama does not show up even for large values of h and α. Nevertheless, in general, bounds to the value of q would appear: $0 < q_0 \leq q \leq 1$. This bound comes from the Poisson equation, $\nabla^2\phi = 4\pi G\rho$ with $\rho > 0$.

One should remark that $\mathcal{K}$ given by (11.8) was obtained neglecting high order terms, assuming quasiperiodicity in the unperturbed motion and averaging to zero the oscillating part. Thus the pendulum model is a rough first approximation to the dynamics and other perturbing terms should be present. However, the main effect of perturbations to the pendulum is to distort somehow the invariant curves and to give rise to a stochastic layer around the separatrix. That is, box and loop should be actually separated by a stochastic layer instead of a separatrix. The larger the strength of the perturbation, the larger is the width of the layer (see the forthcoming chapters).

The derivation given above for $\mathcal{K}$ is a justification of the invariant introduced ad-hoc to compute certain models of elliptical galaxies that respect a third integral. Indeed, if the potential has the form

$$\phi(\boldsymbol{r}) = \psi(r) + \frac{\chi(\theta)}{r^2}, \tag{11.14}$$

with ψ and χ arbitrary functions, then a third integral exists

$$I_3 = \frac{p_\theta^2}{2} - \chi(\theta).$$

The form (11.14) is a particular case of a more general type of potentials introduced almost one century ago by Eddington to study oblate distributions where the ellipsoidal velocity law is exactly satisfied (he showed, however, that the latter condition does not hold if in (11.14) $\chi \neq 0$). Later on, this model was adopted, for instance, by Lynden–Bell in his investigations on statistical mechanics of violent relaxation in rotating elliptical systems. As was pointed out by Eddington, Lynden–Bell and others, (11.14) is unsuitable for any galactic potential so, in general, the third integral for a more realistic model is supposed to be

$$I_3 = \frac{p_\theta^2}{2} - \xi(r, \theta),$$

where $\xi(r,\theta)$ is such that I_3 should satisfy approximately the collisionless Boltzmann equation. No other explicit integral can be expected for a general potential of the form $\phi(r,\theta) = \psi(r) + Q(r)\chi(\theta)$.

Note, however, that for a bar-like galaxy the multipolar expansion of $\phi(\boldsymbol{r})$ has as dominant terms

$$\phi(\boldsymbol{r}) \approx \psi(r) + Q(r)P_2(\cos\theta) \equiv \psi_1(r) + Q_1(r)\cos 2\theta,$$

where $P_2(\mu)$ is the Legendre polynomial of degree 2. If ψ and Q are regular at $r = 0$ we recover the Hamiltonian (11.3) taken as a model for the above discussion.